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FOUNDATIONS

Ketonen, Oiva. Untersuchungen zum Prädikatenkalkül. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 23, 71 pp. (1944). [MF 16495]

Der Verfasser betrachtet Systeme von Axiomen, welche aus endlich vielen Relationszeichen mit Individuenvariabeln mittels der logischen Verknüpfungen und Quantoren gebildet sind. Im Anschluss an die Gentzensche Formalisierung der Logik gibt er für Herleitungen aus einem solchen Axiomensystem eine Normalform an, welche sich besonders gut eignet zur Untersuchung der Herleitbarkeit einer gegebenen Formel. Anwendungen: (1) Widerspruchsfreiheit der Verknüpfungsaxiome der ebenen projektiven Geometrie; sehr einfach ergibt sich die Nichtherleitbarkeit des Desarguesschen und des Pascalschen Satzes; (2) Widerspruchsfreiheit der Verknüpfungsaxiome der ebenen Euklidischen Geometrie; Unabhängigkeit des Parallelenaxioms. *A. Heyting.*

Barcan, Ruth C. A functional calculus of first order based on strict implication. J. Symbolic Logic 11, 1–16 (1946).

The author extends the Lewis systems of modal sentential calculus, S2 and S4, to include quantification with respect to individuals. The axioms assumed in addition to those given by Lewis appear plausible, but no attempt is made to give them a more precise justification. It is shown that the axioms given enable one to show, for the extension of S4, that, if δ results from γ by replacing α by β, and if $X_1, \cdots, X_n$ are all the variables which occur free in α and β, then $(X_1)(X_2)\cdots(X_n)(\alpha\equiv\beta) \dashv (\gamma\equiv\delta)$ is provable. An analogous weaker theorem is provable in the extension of S2.

The conditions given in the rule of inference IV appear to be inadequate. It should apparently also be required that α is not free in A at any place where β is bound.

J. C. C. McKinsey (Stillwater, Okla.).

Hoff-Hansen, Einar. A mathematical interpretation of the classical propositional calculus. Norsk Mat. Tidsskr. 25, 6–12 (1943). (Norwegian) [MF 16985]

The author's interpretation is in terms of ordinary polynomials with rational integral coefficients not reduced modulo 2. These polynomials are associated with expressions of propositional algebra (which is formulated as in Principia Mathematica) by the following conventions: the variables $p, q, \cdots$ of propositional algebra are associated with the indeterminates $x, y, \cdots$ of the polynomials; if the formulas P and Q of propositional algebra are associated with polynomials X and Y, respectively, then $P \vee Q$ is associated with $X\cdot Y$ and $-P$ (not-P) with $1-X$. The polynomials are reduced by replacing higher powers of the indeterminates by their first powers. The author then proves a number of theorems which are equivalent to the following. (1) A polynomial X associated with a formula P takes values belonging to the unit interval (that is, $0\leqq X\leqq 1$) whenever all its indeterminates do. (2) Conversely, if X is any polynomial having the properties just stated, then X is associated with a formula P and any two such P's are equivalent in the propositional algebra. (3) Any such X has, in addition, the property that $X^2=X$ whenever all the indeterminates are given values in the unit interval. (4) If X is associated with an asserted P, then the reduced form of X is identically zero and vice versa. (5) If X is the reduced form of a polynomial associated with a formula P, and if all the indeterminates in X are given the value $\frac{1}{2}$, then $2^n X$ is the number of major terms whose conjunction constitutes the conjunctive normal form of P. (6) If X, Y are all reduced forms of polynomials associated with formulas, then $X\leqq Y$ (for all values of the indeterminates in the unit interval) if and only the reduced form of XY is the same as X; and $X+Y$ is also a reduced form corresponding to a formula if and only if the reduced form of XY is identically zero. The paper is considerably clarified by the note of Skolem reviewed below.

H. B. Curry (State College, Pa.).

Skolem, Th. Some remarks on the preceding article of E. Hoff-Hansen. Norsk Mat. Tidsskr. 25, 13–16 (1943). (Norwegian) [MF 16883]

The author points out that the reduced form of a polynomial, as considered in the paper reviewed above, is its residue with respect to the polynomial ideal generated by x^2-x, y^2-y, z^2-z, etc. He derives the essential theorems of Hoff-Hansen's paper on that basis. Some of the lemmas used may be of interest as algebraic theorems in their own right. These may be stated as follows. Let L denote polynomials of at most the first degree in each variable, with rational coefficients; L' those with integral coefficients; I the unit interval $0\leqq x\leqq 1$; J (J') the elements of L (L') which have values in I whenever their variables have values in I. Then the lemmas are: (1) a polynomial in L vanishes identically if it always vanishes when its variables are given the values 0, 1 in all possible ways; (2) every polynomial with rational coefficients is congruent with respect to the ideal mentioned to one and only one polynomial in L; (3) an element in L is in J if and only if it always has a value in I whenever its variables are given the values 0, 1; (4) a polynomial in L' is in J' if and only if it always takes values 0 or 1 under the same circumstances; (5) if X is in J', then X^2 is congruent to X. These lemmas put the principal theorems of Hoff-Hansen on a more elegant basis.

H. B. Curry (State College, Pa.).

Skolem, Th. Recursive arithmetic. Norsk Mat. Tidsskr 28, 1–12 (1946). (Norwegian) [MF 16884]

This is a general exposition of recursive arithmetic as begun by the author [Skr. Norske Vid. Selsk. Oslo 1923, no. 6] and continued by Gödel, Herbrand, Péter, Kleene, Turing and others. Inasmuch as the author was a pioneer in this field, there is some interest in the way he summarizes the developments and in various remarks which he makes; but the paper contains no essentially new results.

H. B. Curry (State College, Pa.).

Germansky, Baruch. Axioms of the natural numbers. Riveon Lematematika 1, 13 (1946). (Hebrew)

An axiomatization of the set of natural numbers is proposed based on the following two primitive concepts: "x is a natural number" (written $x\epsilon G$) and "x is a neighbor of y" (written xRy). The axioms are (I) $1\epsilon G$, (II) $1Ru$ has exactly one solution $u\epsilon G$, (III) if $x\epsilon G$ and $x\neq 1$ then xRu has exactly two solutions $u_1, u_2\epsilon G$, (IV) xRy implies $x\neq y$, (V) xRy implies yRx, (VI) if G is decomposed into two nonempty disjoint sets A and B then xRy for some $x\epsilon A$ and $y\epsilon B$.

S. Eilenberg (Bloomington, Ind.).

Bing, Kurt. On B. Germansky's axiomatic systems for the foundations of the natural numbers. Riveon Lematematika 1, 21–28 (1946). (Hebrew)

The author proves that the axioms of Germansky [see the preceding review] are equivalent to the usual Peano axioms for the natural numbers. The paper is to be continued.

S. Eilenberg (Bloomington, Ind.).

Chisini, O. Discorso sull'uguaglianza. Rend. Sem. Mat. Fis. Milano 14, 68–80 (1940).

The paper presents in informal fashion some considerations on the empirical and mathematical aspects of the concept of equality. *L. M. Blumenthal* (Columbia, Mo.).

Scholz, Heinrich. Was will die formalisierte Grundlagenforschung? Deutsche Math. 7, 206–248 (1943).

Churchman, C. West. Discussion: Carnap's "On inductive logic." Philos. Sci. 13, 339–342 (1946).

The papers in question appeared in Philos. Sci. 12, 72–97 (1945); Philos. and Phenomenol. Res. 5, 513–532 (1945); these Rev. 7, 46, 189.

Boyer, Carl B. Proportion, equation, function: three steps in the development of a concept. Scripta Math. 12, 5–13 (1946).

Bays, Séverin. Les concepts mathématiques sont-ils inventés ou découverts? Actes Soc. Helv. Sci. Nat. 125, 9–26 (1946).

Levi, B. Physical magnitudes and dimensions. Math. Notae 6, 1–39 (1946). (Spanish)

The paper is a detailed analysis of the role that the concepts of unit and dimension play in mathematical representation of physical phenomena. *I. Opatowski.*

ALGEBRA

Erdös, Paul, and Kaplansky, Irving. Sequences of plus and minus. Scripta Math. 12, 73–75 (1946).

The number of arrangements on a line of m plus ones and n minus ones such that all sums of $1, 2, \cdots, m+n$ terms of the arrangement regarded as a series are at least $m-n$ is shown to be 0, $m>n+1$; $(n+1)^{-1}{}_{2n}C_n$, $m=n+1$; $(n+1)^{-1}(n+m-1)_{m+n}C_m$, $m<n+1$. These numbers also appear in the more general case of all sums at least $m-n-a$.

J. Riordan (New York, N. Y.).

Tietze, Heinrich. Über gewisse Umordnungen von Permutationen. I. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1943, 131–134 (1944).

This first of a series of four papers [see the reviews below] is a collection of definitions and preliminary material best explained as a game of patience. The deck is like the usual one except that the numbers k of kinds and n_i of cards of each kind, $i=1$ to k, are arbitrary. The cards are dealt into packs containing cards of like kind in natural (falling) order until the $(k+1)$th pack is about to be dealt. The cards dealt are then picked up by packs in the reverse order of the dealing and placed at the bottom of the deck, each pack being put into the deck with its cards in natural order. The process is then repeated. After i iterations the deck originally P^0, say, becomes P^i. To each P^i there is a number $s(P^i)$ of "straights" (Verbände), that is, cards of like kind in natural order; $s(P^i)$ is monotone decreasing, since the process tends to collect straights by dealing, reversing the order of pack pick-up and putting packs on the bottom of the deck. Evidently the limit σ of s is not less than k. If $s(P^0)=\sigma$, P is "finite"; for any P there is a value m such that P^m is finite (take m as the least such value). If $P^\rho=P$ for some ρ, P is stable. If $s(P^0)=\sigma=k$, P^0 contains cards of like kind together in natural order and is said to be closed and is excluded from the class of stable P's.

J. Riordan (New York, N. Y.).

Tietze, Heinrich. Über gewisse Umordnungen von Permutationen. II. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1943, 135–148 (1944).

The sequence of permutations P^i obtained by a process described in the paper reviewed above tends either to a closed permutation or a stable (periodic) permutation. This paper gives a criterion for stability. Attention is fixed on a trace element and the numbers $i_1, i_2, \cdots$ of the permutations in which it is among the cards dealt out, along with numbers $l_1, l_2, \cdots$ of the packs into which it is dealt at these appearances. These determine numbers τ_j defined by $\tau_j=[(E-1)i_j-1]k+(E+1)l_j-1$, where E is a shift operator: $Ei_j=i_{j+1}$. If $\tau_{j-1}=\tau_j=qk+r$, the permutation may be stable with period $\pi(kq+r)$, where $\pi(kq+r)=q(q+1)$ if $r=1$, q odd or $r=k-1$, q even; $\pi(kq+r)=q$ if $r=0$, $k=1$; $\pi(kq+r)=2q$ if $r=0$, $k>1$; $\pi(kq+r)=2q(q+1)$ otherwise. If it is stable, then π iterations later the same conditions should be obtained, or else a new period π may be computed and the process repeated. Conditions are given for the period of appearance of the trace element to shorten the test by considering only such appearances. Repetition of the test with another trace element may be needed to determine the minimum i such that P^i is stable. *J. Riordan.*

Tietze, Heinrich. Über gewisse Umordnungen von Permutationen. III. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1943, 269–279 (1944).

Examples are given of the application of the stability criterion developed in part II [see the preceding review]. The first of these is to a shuffle of the usual deck of cards: $k=4$, $n_1=n_2=n_3=n_4=13$, for which P^i is periodic with period 8 for $i=699$. This shuffle is highly selected. For a single instance of random shuffle a period of 20 with $i=61$ is given. Data are also given on 10 random shuffles each of the same deck, and of two other decks, one with 4 kinds each of 4 cards, the other with 2 kinds each of 13 cards.

J. Riordan (New York, N. Y.).

Tietze, Heinrich. Über gewisse Umordnungen von Permutationen. IV. Wahrscheinlichkeitsfragen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1943**, 281–293 (1944).

The number of permutations of elements 1 to n (of one kind) with m straights, or successions of numbers in falling natural order like 5432, is determined as $\binom{n-1}{m-1}(D_m+D_{m-1})$, where $D_n=\Delta^n 0!$ is a rencontres or subfactorial number. [The proof may be shortened by noting that D_n+D_{n-1} is the number of permutations of n elements without successions 21, 32, $\cdots$, n $n-1$, as given by Whitworth, Choice and Chance, Cambridge, 1901, p. 103.] This number is divided into two classes, stable and unstable permutations, enumerated by $\binom{n-1}{m-1}mD_m$ and $\binom{n-1}{m-1}(m-1)D_{m-2}$. Indications are given of the generalizations of this to permutations of n elements of k kinds with n_i elements of the ith kind.

J. Riordan (New York, N. Y.).

Tietze, Heinrich. Über gewisse Umordnungen von Permutationen und ein zugehöriges Stabilitäts-Kriterium. I. Jber. Deutsch. Math. Verein. **53**, 147–212 (1943).

This is the first part of a more detailed presentation of the material of the four papers reviewed above.

J. Riordan (New York, N. Y.).

Nandi, Hari Kinkar. Enumeration of non-isomorphic solutions of balanced incomplete block designs. Sankhyā **7**, 305–312 (1946).

The author enumerates the balanced incomplete block designs with parameters (A): $v=6$, $b=10$, $r=5$, $k=3$, $\lambda=2$, (B): $v=10$, $b=15$, $r=6$, $k=4$, $\lambda=2$. Considering isomorphic solutions as equal, there are only one solution for the design (A) and three for the design (B), all of which may be obtained by block section from the designs $v=b=11$, $r=k=5$, $\lambda=2$ and $v=b=16$, $r=k=6$, $\lambda=2$, respectively. To simplify his enumeration the author proves the following theorem. If x_i denotes the number of varieties common to a given block and the ith of the remaining blocks and f_α the frequency with which x_i takes the value α, then the f_α are unique if either $\lambda=1$ or the design is obtained by block section and $\lambda=2$.

H. B. Mann (Columbus, Ohio).

Nandi, Harikinkar. A further note on non-isomorphic solutions of incomplete block designs. Sankhyā **7**, 313–316 (1946).

The author enumerates all nonisomorphic solutions of the incomplete balanced block design with $v=b=15$, $r=k=7$, $\lambda=3$ and its derived and residual configurations $v=7$, $b=14$, $r=6$, $k=3$, $\lambda=2$ and $v=8$, $b=14$, $r=7$, $k=4$, $\lambda=3$, respectively. He finds five nonisomorphic solutions of the symmetrical design and four each of the derived and residual configuration.

H. B. Mann (Columbus, Ohio).

Husain, Q. M. Impossibility of the symmetrical incomplete block design with $\lambda=2$, $k=7$. Sankhyā **7**, 317–322 (1946).

The author proves by tactical enumeration that the balanced incomplete block design with parameters $v=b=22$, $r=k=7$, $\lambda=2$ is impossible.

H. B. Mann.

Bhattacharya, K. N. A new solution in symmetrical balanced incomplete block designs ($v=b=31$, $r=k=10$, $\lambda=3$). Sankhyā **7**, 423–424 (1946).

The author gives the first solution for the design mentioned in the title. The solution is obtained by a modification of the methods developed by Bose [Ann. Eugenics **9**, 358–399 (1939); Bull. Calcutta Math. Soc. **34**, 17–31 (1942); these Rev. **4**, 33; **5**, 87]. The design 21, 30, 10, 7, 3 can easily be obtained from the new design by the process of block section.

H. B. Mann (Columbus, Ohio).

Iglisch, Rudolf. Über den Fundamentalsatz der Algebra. Deutsche Math. **5**, 339–340 (1940). [MF 14339]

In order to prove that a polynomial $f(z)=a_0+a_1z+\cdots+a_nz^n$ has at least one zero the author uses partial differentiation with respect to the two real variables x and y ($z=x+iy$). By a discussion of the second order derivatives he shows that at the minimum of $|f(z)|^2$ the polynomial is zero.

A. C. Schaeffer (Stanford University, Calif.).

Onicescu, O. Sur le théorème fondamental de l'algèbre. Mathematica, Timişoara **22**, 208–214 (1946).

Proof of the fundamental theorem in the form "a polynomial with real coefficients and of even degree has a real quadratic factor," avoiding the use of complex numbers. The work starts with a lemma (which may be generalized considerably) of which the proof is simple: necessary and sufficient conditions for $a_0x^{4m+2}+\cdots+a_{4m+2}$ to have a quadratic factor $x^2-\delta$ are identical with necessary and sufficient conditions for $a_0x^{2m+1}+a_2x^{2m}+\cdots+a_{4m}x+a_{4m+2}$ and $a_1x^{2m}+a_3x^{2m-1}+\cdots+a_{4m-1}x+a_{4m+1}$ to have a common factor $x-\delta$. By applying this lemma and using purely algebraic reasoning, it is proved that a polynomial of degree $4m+2$ with real coefficients has a real quadratic factor. From this point on, methods of analysis are applied (mapping of one real plane on to another real plane by an algebraic function and topological and continuity considerations).

A. J. Kempner (Boulder, Colo.).

Conte, Luigi. La limitazione delle radici reali di una equazione algebrica secondo Newton. Atti Mem. Accad. Sci. Padova. Mem. Cl. Sci. Fis.-Mat. (N.S.) **58**, 163–174 (1942).

Limits for the absolute values of the roots of an algebraic equation are studied. The methods discussed were given by Newton.

E. Lukacs (Cincinnati, Ohio).

Batschelet, Eduard. Über die Abschätzung der Wurzeln algebraischer Gleichungen. Elemente der Math. **1**, 73–81 (1946).

This paper deals essentially with separation of roots of algebraic equations with real or complex coefficients. The main tool used is Grace's Faltungssatz: given $f(x)=x^n+\binom{n}{1}a_1x^{n-1}+\binom{n}{2}a_2x^{n-2}+\cdots+a_n$, $g(x)=b_0x^n+b_1x^{n-1}+b_2x^{n-2}+\cdots+b_n$, with $b_n-a_1b_{n-1}+a_2b_{n-2}-\cdots\pm a_nb_0=0$ ($f(x)$, $g(x)$ apolar); then every circular region in the plane of complex numbers which contains all roots of one of the equations $f(x)=0$, $g(x)=0$, contains at least one root of the other equation. The author derives results which had been obtained by different methods by Newton, Laguerre, Birkhoff and Jensen, Fekete, and Takahashi, and extends several of them.

We mention the following: (I) An upper bound for the error committed if the Newton approximation method is broken off after k steps. Let $Z=\lim_{k\to\infty}x_k$, $x_{k+1}=x_k+h_k$, $h_k=-f(x_k)/f'(x_k)$, be the root approximated, n the degree of the equation; then $|Z-x_k-(nk_k)/2|<|h_k|n/2$. (II) For $f(x)$ as above, at least one root of $f(x)=0$ lies inside, or on the circumference of, any circle through the points $\pm(a_n/a_{n-2})^{1/n}$ [similar to results by Laguerre]. (III) If

$\lambda=\{|f(\alpha)|/|f(\beta)|\}^{1/n}$, α,β any complex numbers, at least one root of $f(x)=0$ lies inside or on the circle $|x-\alpha|=\lambda|x-\beta|$ [extension of a result by Fekete]. (IV) With $f(x)$ as above, assume that all roots of $f(x)=0$ lie inside a circle K for which $x=0$ lies outside; then all points $-a_k/a_{k-1}$ $(k=1,\cdots,n)$ lie inside K [extension of a result by Takahashi]. The results depend mainly on judicious selection of the polynomial $g(x)$. *A. J. Kempner* (Boulder, Colo.).

Anghelutza, Th. Sur la détermination de l'indice d'une fonction rationnelle. Mathematica, Timişoara **22**, 41–50 (1946).

Let $U=a_0x^m+a_1x^{m-1}+\cdots+a_m$, $V=b_0x^p+b_1x^{p-1}+\cdots+b_p$ be real polynomials with $p\leqq m$ and with no zeros in common. In order to determine the Cauchy index I of the rational function V/U by use of the Sturm sequence $U, V, V_1, \cdots, V_p$, the author shows that the term of highest degree in V_q is essentially

$$C_q(x)=(-1)^{\frac{1}{2}q(q+1)}b_0^{-(m-p+1)}x^{p-q}\begin{vmatrix} b_0 & b_1 & \cdots & b_{m-p+2q-2} & b_{m-p+2q-1}\\ 0 & b_0 & \cdots & b_{m-p+2q-3} & b_{m-p+2q-2}\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \cdots & b_{q-1} & b_q\\ a_0 & a_1 & \cdots & a_{m-p+2q-2} & a_{m-p+2q-1}\\ 0 & a_0 & \cdots & a_{m-p+2q-3} & a_{m-p+2q-2}\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \cdots & a_{q-1} & a_q \end{vmatrix},$$

where $a_k=0$ if $k<0$ or if $m<k$, and $b_k=0$ if $k<0$ or $p<k$. Thus $I=V_{+\infty}-V_{-\infty}$, where $V_{+\infty}$ and $V_{-\infty}$ denote the number of variations of sign in the sequence $a_0x^m, b_0x^p, C_1(x), \cdots, C_p(x)$ for $x=+\infty$ and $x=-\infty$, respectively. Finally, this result is applied to a derivation of (1) Hermite's theorem that I is the difference between the number of zeros of $U+iV$ having negative imaginary parts and those having positive imaginary parts; (2) Hurwitz's criterion that all the zeros of a polynomial have negative real parts. *M. Marden.*

Morrison, I. F. The solution of three-term simultaneous linear equations by the use of submatrices. Engineering J. **29**, 80–83 (1946).

Couffignal, Louis. Recherches de mathématiques utilisables. La résolution numérique des systèmes d'équations linéaires. I. L'opération fondamentale de réduction d'un tableau. Revue Sci. (Rev. Rose Illus.) **82**, 67–78 (1944).

This memoir consists of a detailed discussion of simultaneous linear equations starting from the very beginning with definitions of terms, explanation of notation and theorems. The author manages to avoid the use of the term "matrix," a rectangular array of numbers being called a "table." The new table obtained after one variable has been eliminated is called a "reduced table" and rules for "reducing" a table are given as well as numerous theorems pertaining to "reduction," "rank," linear independence, etc. Practical applications of the principles are to be treated in the next memoir. *W. E. Milne* (Corvallis, Ore.).

Loomis, Lynn H. On a theorem of von Neumann. Proc. Nat. Acad. Sci. U. S. A. **32**, 213–215 (1946).

The author gives an elementary proof of the following result of J. von Neumann. Let (a_{ij}) and (b_{ij}) be two $n\times m$ matrices such that $a_{ij}>0$ for all i, j. Then there exists a unique λ, a vector $x=(x_1, \cdots, x_m)$ with $x_j\geqq 0$ and $\sum x_j=1$, and a vector $y=(y_1, \cdots, y_n)$ with $y_i\geqq 0$ and $\sum y_i=1$, such that

$$\lambda\sum_{j=1}^{m}a_{ij}x_j\geqq\sum_{j=1}^{m}b_{ij}x_j, \qquad i=1, \cdots, n,$$

$$\lambda\sum_{i=1}^{n}a_{ij}y_i\leqq\sum_{i=1}^{n}b_{ij}y_i, \qquad j=1, \cdots, m.$$

N. H. McCoy (Northampton, Mass.).

Brusotti, Luigi. Dimostrazione di un lemma algebrico utile in questioni di analisi. Ann. Scuola Norm. Super. Pisa (2) **11**, 211–215 (1942). [MF 16761]

Given the polynomials $f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n)$, suppose $a_i<b_i$, $i=1, \cdots, n$, and (I) $f_h<0$ for $x_h=a_h$, $f_h>0$ for $x_h=b_h$. Suppose (II) the solutions, real and complex, of each of the systems $f_1=0, \cdots, f_h=0, x_{h+1}=a_{h+1}, \cdots, x_n=a_n$ for $h=1, \cdots, n$, are finite in number and simple "in the usual sense of algebra." Then the number of points at which all the f_i vanish in the region R: $a_i\leqq x_i\leqq b_i$, $i=1, \cdots, n$, is odd. The proof uses mathematical induction. As a corollary, the author shows that under condition (I) (without imposing (II)), there must be at least one point in R at which all the f_i vanish. [This last result is a known theorem of L. E. J. Brouwer, for the more general case that the f_i are assumed merely to be continuous functions, not necessarily polynomials.] *A. B. Brown* (Flushing, N. Y.).

Parodi, Maurice. Sur l'existence des réseaux électriques. C. R. Acad. Sci. Paris **223**, 23–25 (1946).

The author proves that, if the elements a_{hk} of a determinant of order n satisfy the relations

$$a_{hh}>0, \quad a_{hh}>\sum_k a_{hk},$$

$$h=1, \cdots, n;\ k=1, \cdots, h-1, h+1, \cdots, n,$$

the determinant is positive. This theorem has applications to determinants occurring in the theory of electrical networks. *J. Williamson* (Flushing, N. Y.).

Williamson, John. Determinants whose elements are 0 and 1. Amer. Math. Monthly **53**, 427–434 (1946).

The maximum determinant of a matrix of order 7 whose elements are ± 1 is $2^6 9$ and there is essentially only one such matrix. This is below the maximum of Hadamard. Less complete results are obtained for matrices of orders 9, 10 and 11. *C. C. MacDuffee* (Río Piedras, P. R.).

Turowicz, A. Sur un propriété des déterminants. Ann. Soc. Polon. Math. **18**, 118–122 (1945).

The minimum value of a determinant of order n with real elements a_{ik} such that $|a_{ik}-\delta_{ik}|\leqq a\leqq 1/n$, where $a>0$, is $1-na$. One such determinant with minimum value has the elements $b_{ik}=\delta_{ik}-a$, and every determinant of minimum value satisfying this condition is given by $a_{ik}=b_{ik}\cdot\eta_i\eta_k$, where $\eta_1, \cdots, \eta_n$ are arbitrary real numbers of absolute value 1. *C. C. MacDuffee* (Río Piedras, P. R.).

Calderón Jiménez, Manuel. Determinant of a rectangular matrix. Revista Mat. Hisp.-Amer. (4) **5**, 231–250 (1945). (Spanish) [MF 15884]

The author defines a real function of rectangular arrays of numbers which has many of the properties of the determinant function of square matrices. In an array of one row the adjoint of an element is defined as 1 or -1 according

as the number of the column in which the element appears is odd or even, the development of the array is the sum of the products of each element by its adjoint and the determinant of the array is the number obtained by computing the development. In an array of m rows and n columns $(1<m\leqq n)$ the adjoint of an element is its algebraic complement and a development according to the elements of a row is the sum of the products of each element of the row by its adjoint. Since development is seen to be invariant under a change of rows, the determinant of an array may be defined as the number obtained by computing any row development. The author proves a theorem analogous to Laplace's development and states other properties of the function which are analogous to those enjoyed by ordinary determinants. *L. M. Blumenthal* (Columbia, Mo.).

Tambs Lyche, R. Sur un groupe particulier de transformations linéaires. Norske Vid. Selsk. Forh., Trondhjem **18**, no. 26, 103–105 (1945).

The author notes the existence of the group of third order Lorenzian matrices. *C. C. MacDuffee.*

Garnir, Henri. Sur la détermination des matrices satisfaisant à un système de relations de la théorie du méson. C. R. Acad. Sci. Paris **223**, 539–540 (1946).

The problem is the determination of all systems of n Hermitian matrices satisfying $A_iA_jA_k+A_kA_jA_i=A_i\delta_{jk}+A_k\delta_{ji}$, $i, j, k=1, \cdots, n$. A theorem of complete reducibility is stated and all irreducible systems are constructed from linear operators on p-vectors over a space of $n+1$ dimensions. If A_i satisfy the required relations, so do $B_i^{(p)}=A_iA_p+A_pA_i$, $i\neq p$, $B_p^{(p)}=A_p$. Proofs are not given. *W. Givens.*

Hsu, P. L. On a factorization of pseudo-orthogonal matrices. Quart. J. Math., Oxford Ser. **17**, 162–165 (1946).

A pseudo-orthogonal matrix of order $p+q$ is defined to be a real automorph of the matrix $I_{p,-q}=I_p\dotplus(-I_q)$, where I_p is the unit matrix of order p and $\dotplus$ denotes direct sum; i.e., a real matrix A is pseudo-orthogonal if (1) $AI_{p,-q}A'=I_{p,-q}$, where A' is the transpose of A. The author proves that if A satisfies (1) and $p\leqq q$, then (2) $A=(\Gamma_1\dotplus\Gamma_2)(\Lambda\dotplus I_q)(\Omega_1\dotplus\Omega_2)$, where Γ_1 and Ω_1 are orthogonal matrices of order p and Γ_2 and Ω_2 are orthogonal matrices of order q. In (2),

$$\Lambda=\begin{pmatrix}(1+\lambda_1)^{\frac12}\dotplus\cdots\dotplus(1+\lambda_p)^{\frac12} & \lambda_1^{\frac12}\dotplus\cdots\dotplus\lambda_p^{\frac12}\\ \lambda_1^{\frac12}\dotplus\cdots\dotplus\lambda_p^{\frac12} & (1+\lambda_1)^{\frac12}\dotplus\cdots\dotplus(1+\lambda_p)^{\frac12}\end{pmatrix}$$

and

$$\Omega_2=\begin{pmatrix}\Pi_1 & \Pi_2\\ \Pi_3 & \Pi_4\end{pmatrix},$$

where Π_1 is a p by p matrix and Π_4 a $q-p$ by $q-p$ matrix of triangular type (zeros below the main diagonal). He also shows that this factorization (2) is unique provided that $\lambda_1, \cdots, \lambda_p$ are all positive, distinct and arranged in ascending order and that Π_4 is nonsingular. *J. Williamson.*

Liang, S. L. The polar factorisation of a singular matrix. Acad. Sinica Science Record **1**, 330–331 (1945).

The author proves that, if A is any singular matrix, then $A=UP$, where P is a positive semi-definite Hermitian matrix and U is a unitary matrix. The matrix P is uniquely determined but U is not. This is analogous to the well-known fact that, if A is nonsingular, then $A=UP$, where P is a positive definite Hermitian matrix and U a unitary matrix, both uniquely determined. *N. H. McCoy.*

Petrescu, St. Sur la réduction à une forme canonique d'une forme bilinéaire symétrique gauche, par des transformations orthogonales. Mathematica, Timişoara **22**, 1–12 (1946).

The author determines explicitly all orthogonal matrices O such that

$$OSO'=\begin{bmatrix}0 & a & 0 & 0\\ -a & 0 & 0 & 0\\ 0 & 0 & 0 & b\\ 0 & 0 & -b & 0\end{bmatrix},$$

where S is a four-rowed skew-symmetric matrix.

J. Williamson (Flushing, N. Y.).

Dmitriev, N., and Dynkin, E. On characteristic roots of stochastic matrices. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 167–184 (1946). (Russian. English summary)

Soit $A_n\equiv(a_{ij})_n$ une matrice stochastique d'ordre n. Les a_{ij} représentent les probabilités de passage, en une unité de temps, d'un certain système de l'état E_i à l'état E_j, $i, j=1, 2, \cdots, n$. Donc $a_{ij}\geqq0$, $\sum_{j=1}^n a_{ij}=1$. Les auteurs étudient le problème suivant, posé par Kolmogoroff: pour n donné, déterminer l'ensemble M_n des valeurs caractéristiques λ de l'ensemble des A_n. Or, pour que l'on ait $\lambda\epsilon M_n$ il faut et il suffit qu'un certain polygone convexe de k côtés $(k\leqq n)$, soumis à la transformation définie par le nombre complexe λ, devienne partie (ou totalité) de lui-même. Ceci permet d'utiliser des méthodes géométriques élémentaires. On a, de suite: $M_n\subset C$, cercle de rayon un centré sur l'origine, et $M_n\subset M_{n+1}$; pour obtenir M_n il suffit de trouver ses nombres extremaux λ_e, c'est-à-dire, tels que, si $\lambda_e\epsilon M_n$, alors $\alpha\lambda_e\bar{\epsilon}M_n$ pour tout $\alpha>1$. Une condition nécessaire est fournie par le théorème fondamental qui, en langage de probabilités, s'énonce ainsi. Soit $\lambda_e\epsilon M_n-M_{n-1}$, dont l'argument est compris entre $2\pi p/n$ et $2\pi(p+1)/n$, extremités comprises. Si λ_e est un λ de A_n alors, en numérotant convenablement les états, le système ne peut passer, en une unité de temps, d'un état E_i que dans les états E_{i+p} et E_{i+p+1}.

Un polygone Q sera dit cyclique, engendré par μ, s'il est l'enveloppe convexe des affixes $1, \mu, \mu^2, \cdots$; M_n contient tous les Q de k côtés $(k\leqq n)$. Les auteurs emettent l'hypothèse que M_n ne contient pas d'autres points et la prouvent pour $n\leqq5$. Tous les résultats obtenus sont étendus à des matrices quelconques B_n à éléments non-négatifs, grâce au théorème suivant, communiqué par Kolmogoroff: une matrice indécomposable B_n a les mêmes valeurs caractéristiques λ qu'une matrice ρA_n, où ρ est le maximum du module des λ et A_n une matrice stochastique. *M. Loève* (Londres).

Todd, J. A. The complete system of the binary (3, 1) form. Proc. Cambridge Philos. Soc. **42**, 196–205 (1946).

The author finds by traditional methods the complete system of irreducible concomitants of a double binary form of degrees 3, 1 in the respective variables. The system contains 26 forms which range from degree 1 to degree 9 in the groundform, with one more form of degree 12.

D. E. Littlewood (Swansea).

Todd, J. A. The complete system of the binary (4, 1) form. Proc. Cambridge Philos. Soc. **42**, 206–216 (1946).

The author finds by traditional methods the complete system of irreducible concomitants of a double binary form of degrees 4, 1 in the respective variables. The system contains 63 forms which range from degree 1 to degree 14 in the

groundform, with two more forms of degrees 16 and 18, respectively. *D. E. Littlewood* (Swansea).

Todd, J. A. A note on ternary-binary forms. J. London Math. Soc. **20**, 209–213 (1945).

The author considers double forms $f=A_x{}^m a_y{}^n$ of order m in a ternary variable $x=\{x_1, x_2, x_3\}$ and of order n in a binary variable $y=\{y_1, y_2\}$. Independent linear transformations are imposed upon x and y, respectively, and the consequent invariant theory is considered. The corresponding theory of double binary forms has been treated by Peano and others, but little is known of the present forms. They have considerable geometrical interest.

From a knowledge of the types of concomitants of a ternary m-ic, that of the present (m, n) form follows as a further problem in purely binary forms. But such types are only known for the cases $m=1, 2$. In the present note, after establishing the above procedure, the author works out the (1, 1) and the (1, 2) forms in detail. Geometrically the (1, 1) form answers to a coplanar pencil of lines, and has two concomitants: itself and the form $(ABu)(ab)$ which represents the vertex of the pencil, x giving point coordinates and u line coordinates.

The complete system of the (1, 2) form consists of six forms and is interpreted by a conic and various lines and points. The (1, 3) form would involve a knowledge of the binary system belonging to one quartic and two cubics.
H. W. Turnbull (St. Andrews).

Turnbull, H. W. The critical concomitant of bilinear forms. Proc. London Math. Soc. (2) **49**, 99–127 (1946).

The author considers the projective concomitants of the bilinear form $\phi=\sum_{i,j=1}^n u_i a_{ij} x_j = u'Ax$, where A is the matrix (a_{ij}) and $u=\{u_1, \cdots, u_n\}$ and $x=\{x_1, \cdots, x_n\}$ are two sets of contragredient variables. The complete system of concomitants requires several sets of variables $v, w, \cdots$ cogredient to u and several sets $y, z, \cdots$ cogredient to x and consists, as previously shown by the author, of five types [same Proc. (2) **33**, 1–21 (1931)].

If x is a point of the $(n-1)$-space and u is a prime, Ax denotes the image point of x and $A'u$ the image prime of u in the collineation whose matrix is A and whose equation is $u'Ax=0$. The author proves that the whole system of concomitants can be expressed in terms of invariants and forms of the type $u_x=\sum_{i=1}^n u_i x_i$, $(uv \cdots w)$ and $(xy \cdots z)$, where $(uv \cdots w)$ and $(xy \cdots z)$ are determinants of order n whose columns are the sets $u, v, \cdots, w$ and $x, y, \cdots, z$, respectively. However, now any $x, y, \cdots, z$ may be an original point or an image point and any $u, v, \cdots, w$ an original prime or an image prime. If $A^i x = x^i$, $i=0, 1, \cdots$, certain concomitants, called Q' concomitants, have the form $(x^i x^j \cdots y^k y^l \cdots)$ and a Q' concomitant $Q_s'=(X^{p_1} Y^{p_2} \cdots Z^{p_h})$, $p_1+\cdots+p_h+h=n$, $p_1 \geqq \cdots \geqq p_h$, where $X^r = x^0 x^1 x^2 \cdots x^r$, is called a standard form. The standard forms may be arranged in order, so that Q_s' precedes $Q_t'=(X^{q_1} Y^{q_2} \cdots W^{q_h})$ if $p_i=q_i$, $i=1, \cdots, m$, $p_{m+1}>q_{m+1}$. The first concomitant Q_s' which does not vanish identically is called the critical form. The h indices p_i+1 of this form are the degrees of the h invariant factors of the matrix A. This critical form is a multilinear form in several sets of compound coordinates and it is shown that it can be factored into linear factors where the exponents of the distinct linear factors determine the exponents of the elementary divisors of A. In particular, if the minimal equation of A is of degree n so that A has a single invariant factor with k distinct elementary divisors $(\lambda-\lambda_i)^{e_i}$, $i=1, \cdots, k$, the critical concomitant, which in this case is an n-ary n-ic covariant in x, has the factors $\xi_i{}^{e_i}$, $i=1, \cdots, k$, where $\xi_1, \cdots, \xi_k$ are k linearly independent linear forms in $x_1, \cdots, x_n$.

Since any collineation may be constructed by successive polarizations in two quadrics, the author proceeds to show, by slight modifications of his results, the existence of a critical concomitant for two quadratic forms in n variables and to determine from its factors the exponents of the elementary divisors of the pencil formed by the symmetric matrices of the two quadratic forms. The existence of a critical covariant in the comparatively simple case that the two symmetric matrices have a single invariant factor had previously been noted [J. Williamson, Amer. J. Math. **56**, 339–348 (1934)].

The methods of proof are those of symbolic invariant theory and, while each theorem is proved for a general value of n, illustrations in most cases are also worked out in detail for specific small values of n. *J. Williamson.*

Molenaar, P. G. Primitive-symmetric projective invariants. Nederl. Akad. Wetensch., Proc. **49**, 238–250 = Indagationes Math. **8**, 135–147 (1946). (Dutch) [MF 16578]

Molenaar, P. G. Primitive-symmetric projective invariants. II. Nederl. Akad. Wetensch., Proc. **49**, 357–368 = Indagationes Math. **8**, 226–237 (1946). (Dutch) [MF 16584]

Molenaar, P. G. Primitive-symmetric projective invariants. III. Nederl. Akad. Wetensch., Proc. **49**, 470–478 = Indagationes Math. **8**, 325–333 (1946). (Dutch) [MF 16830]

These papers form three parts of one article. The isomers $I_E, I_A, \cdots, I_G$ of the simultaneous invariant I_E of the ground forms $a_x{}^p, \cdots, l_x{}^p$ are obtained by applying to I_E the $n=t!$ permutations $E, A, \cdots, G$ of the symmetric group $\mathfrak{S}_t$. Let $I=I_E e_E + I_A e_A + \cdots + I_G e_G$ be the representative vector of I_E in the group algebra. Then [cf. A. Young, Proc. London Math. Soc. (2) **28**, 255–292 (1928)]

$$I=\sum_{i,q,\sigma} \frac{n}{n_\sigma} L\overset{i}{\underset{q}{(\sigma)}}\,\zeta\overset{q}{\underset{i}{(\sigma)}},$$

where the irreducible representation σ of $\mathfrak{S}_t$ is spanned by the basic elements $\zeta\overset{q}{\underset{i}{(\sigma)}}$, $q, i=1, \cdots, n_\sigma$, and where (Kronecker δ) $\zeta\overset{l}{\underset{i}{(\sigma)}}\zeta\overset{k}{\underset{s}{(\rho)}}=\delta(\sigma)(\rho)\delta^l_s\,\zeta\overset{k}{\underset{i}{(\sigma)}}$. The functions $L\overset{i}{\underset{q}{(\sigma)}}$ being linear functions of the isomers of I_E (and vice versa) are themselves invariants of the given ground forms and are called the primitive symmetric invariants associated with I relative to the basis $\zeta\overset{q}{\underset{i}{(\sigma)}}$. Any symmetry possessed by I_E implies a linear dependence of its isomers and consequently of the associated primitive symmetric invariants. Such a relation $\sum_{i,q,\sigma} p\overset{q}{\underset{i}{(\sigma)}} L\overset{i}{\underset{q}{(\sigma)}}=0$ leads to the component relations $\sum_i p\overset{q}{\underset{i}{(\sigma)}} L\overset{i}{\underset{q}{(\sigma)}}=0$ and from these relations a module-basis for the isomers of I_E can be found. Thus far the paper is largely a clarification of the work of A. Young [loc. cit. and other papers]. The theory is applied to the symbolic invariant $(ab)^2(cd)^2(ad)(bc)$ of the ground forms $a_x{}^3, b_x{}^3, c_x{}^3, d_x{}^3$, both when these are distinct and equivalent cubics. Interpretations of the resulting primitive symmetric invariants are given.

A proof is given that an invariant of the mixed ground form

$$F=\sum A^{i_1, \cdots, i_p, k_1, \cdots, k_r} x_{i_1} \cdots x_{i_p} \xi_{k_1} \cdots \xi_{k_r} = f\phi,$$

where $f=a_x^p$, $\phi=\alpha_\xi^\pi$, is expressible in terms of outer products of the types $(ab\cdots)$, $(\alpha\beta\cdots)$, the type $(a\alpha\cdots)$ being absent. Let L and Λ be invariants linear and homogeneous in the coefficients of t ground forms $f_1=a_x^p$, $f_2=b_x^p$, $\cdots$ and $\phi_1=\alpha_\xi^\pi$, $\phi_2=\beta_\xi^\pi$, $\cdots$, respectively. Then repeated applications of contraction operators to $L\Lambda$ leads to a simultaneous invariant $[L\Lambda]$ of the t ground forms

$$F_1=\sum A^{i_1,\cdots,i_p,k_1,\cdots,k_\pi}x_{i_1}\cdots x_{i_p}\xi_{k_1}\cdots\xi_{k_\pi},$$
$$F_2=\sum B^{i_1,\cdots,i_p,k_1,\cdots,k_\pi}x_{i_1}\cdots x_{i_p}\xi_{k_1}\cdots\xi_{k_\pi},\cdots$$

which is linear in each set of coefficients. The n isomers of L and the n isomers of Λ thus yield by contraction n^2 invariants of the mixed form F. The invariants $[L\binom{i}{k}\Lambda\binom{p}{q}]$ are not in general primitive symmetric, since the direct product of two irreducible representations is not in general irreducible, but methods are described by which these invariants can be expressed in terms of primitive symmetric invariants. Application is made to the case $F=a_x^3\alpha_\xi^3$. Finally, the transvectants of two binary mixed forms are studied. [The paper contains some printer's errors and in one equation in § 6 the symbol L is used in two different senses.]

D. E. Rutherford (St. Andrews).

THEORY OF GROUPS

Piccard, Sophie. Des systèmes de substitutions regulières indépendantes qui engendrent un groupe régulier. Comment. Math. Helv. 19, 134–152 (1946).

Let (A) $S_1\cdots S_m$ $(m>1)$ be an independent set of regular permutations of degree n, let (B) $S_{i_1},\cdots,S_{i_k}$ be a proper subset of $S_1,\cdots,S_m$, let (C) $C_1,\cdots,C_l$ be systems of transitivity of the group $(S_{i_1},\cdots,S_{i_k})$ generated by $S_{i_1},\cdots,S_{i_k}$, and let S_i be a member of (A) but not of (B). (A set (A) is called independent if $(S_1,\cdots,S_m)\neq(S_{i_1},\cdots,S_{i_k})$ for any choice of $i_1,\cdots,i_k$ with $k<m$.) The set (A) is called a regular system if, for all choices of $i,i_1,\cdots,i_k$, (1) $(S_1,\cdots,S_m)$ is transitive; (2) $(S_{i_1},\cdots,S_{i_k})$ is not transitive; (3) S_i maps no letter of any C_j into any letter of C_j; (4) $C_1,\cdots,C_l$ have equal order $u=n/l$; (5) if S_i maps the elements of C_1 into r different C_j then S_i maps the elements of each C_h into r different C_j; (6) $u=rp$ where S_i send either p or 0 elements of C_h into C_j; (7) $(S_{i_1},\cdots,S_{i_k})$ has order u; (8) (a) if any cycle of S_i contains at most one letter of each C_j then no cycle of S_i can contain more than one letter of any C_j, (b) if one cycle $(a_1,\cdots,a_t)$ of S_i contains more than one element of some C_j and if x is the smallest integer such that a_n and a_{n+x} belong to the same set C_j then each system of transitivity of S_i^x is contained in one of the C_h. It is proved that $G=(S_1,\cdots,S_m)$ is regular (for $S_1,\cdots,S_m$ independent) only if (A) is a regular system, and an example is given where (A) is a regular system but G is not regular. If (A) is a regular system, and if for some i the set (B) $S_1,\cdots,S_{i-1}$, $S_{i+1},\cdots,S_m$ and S_i have $r=1$ [cf. (6) above], then G is regular if and only if (I) $S_i^l\epsilon G_1=(S_1,\cdots,S_{i-1},S_{i+1},\cdots,S_m)$ and (II) $S_i^fS_jS_i^{-f}\epsilon G_1$ for all f and all $j\neq i$. Two applications are made to the case in which G is the direct product of the groups $(S_1),\cdots,(S_m)$. *R. M. Thrall* (Ann Arbor, Mich.).

Zappa, Guido. Sulla relazione tra il rango e il tipo di un gruppo. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 574–585 (1941).

The subgroup F of the finite group G is termed a fundamental subgroup of G if there exists an element x in G, and therefore in F, such that F is the centralizer of x in G. Assume the existence of a sequence $G_1<\cdots<G_{i-1}<G_i<\cdots<G_r<G$ of fundamental subgroups of G and denote by n the number of different fundamental subgroups of G. Then the author proves that $8r\leqq n+7$, a result that, at least for $9<n$, improves upon results previously obtained by Cipolla and Scorza. *R. Baer* (Urbana, Ill.).

★Markoff, A. Foundations of the algebraic theory of tresses. Trav. Inst. Math. Stekloff 16, 53 pp. (1945). (Russian. English summary)

The booklet gives a systematic treatment of the theory of tresses from a purely group-theoretic point of view. The group $\mathfrak{Z}_n$ of tresses of order n is defined by generators $\sigma_1,\cdots,\sigma_{n-1}$ and relations

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1},\qquad i=1,\cdots,n-2,$$
$$\sigma_i\sigma_j=\sigma_j\sigma_i,\qquad i<j-1.$$

Let $\mathfrak{N}_n$ denote the least invariant subgroup of $\mathfrak{Z}_n$ containing the elements $\sigma_1^2,\cdots,\sigma_{n-1}^2$. The quotient group $\mathfrak{Z}_n/\mathfrak{N}_n$ is isomorphic with the symmetric group of order n. A system of generators and relations in the group $\mathfrak{N}_n$ is given following W. Burau [Abh. Math. Sem. Hamburgischen Univ. 9, 117–124 (1933)].

Two solutions of the word problem in $\mathfrak{Z}_n$ are developed. One, due to Artin [Abh. Math. Sem. Hamburgischen Univ. 4, 47–72 (1926)], consists in representing $\mathfrak{Z}_n$ faithfully as a group of automorphisms of the free group $\mathfrak{F}_n$ with n generators. The geometric elements of Artin's proof are eliminated. The other solution, due to A. Ivanovsky [unpublished], consists in giving a normal form for the elements of $\mathfrak{Z}_n$.

The group $\mathfrak{N}_n$ has a normal series

$$\mathfrak{N}_n=\mathfrak{N}_{n,1}\supset\mathfrak{N}_{n,2}\supset\cdots\supset\mathfrak{N}_{n,n}=(1),$$

where each factor $\mathfrak{N}_{n,l-1}/\mathfrak{N}_{n,l}$ is a free group with $l-1$ free generators. A connection between the groups $\mathfrak{N}_n$ and $\mathfrak{N}_l$ for $l<n$ is also established. *S. Eilenberg.*

Brauer, Richard. On blocks of characters of groups of finite order. II. Proc. Nat. Acad. Sci. U. S. A. 32, 215–219 (1946).

[Part I appeared in the same Proc. 32, 182–186 (1946); these Rev. 8, 14]. If the modular character φ_ν^i of N_i belongs to a block $\bar{B}_\sigma$ of N_i, then the decomposition numbers $d_{\mu\nu}^i$ can be different from 0 only for ordinary characters of G which belong to the block B_τ of G determined by $\bar{B}_\sigma$. If ρ and σ belong to different sections of G, then $\sum_\mu'\zeta_\mu(\rho)\zeta_\mu(\sigma)=0$, where ζ_μ ranges over all the characters of G belonging to a fixed block B_τ. If B_τ has the defect group D_τ and if no element of D_τ is conjugate to π_i, then $\zeta_\mu(\rho)=0$ for all characters ζ_μ of B_τ and all elements of the section of π_i. If B_τ is of defect d_τ, there exist blocks $\bar{B}_\sigma$ of defect d_τ of N_i which determine B_τ, if and only if π_i is conjugate in G to an invariant element of D_τ. If π_i is conjugate to an invariant element of D_τ, we can choose the block $\bar{B}_\sigma$ of defect d_τ of N_i in such a manner that it determines the block B_τ of G and that for every ζ_μ in B_τ there exists a φ_ν^i in $\bar{B}_\sigma$ such that $d_{\mu\nu}^i\neq0$. If the block $\bar{B}_\sigma$ of N_i determines the block B_τ of G, the defect of $\bar{B}_\sigma$ is at most equal to l, where p^l is the order of a maximal p-subgroup of N_i which is conjugate in G to a subgroup of the defect group D_τ of B_τ. If the degree z_μ of the character ζ_μ of B_τ is not divisible by p^{a-d+l}, where d is the defect of B_τ, there exists a character φ_ν^i of N_i which belongs to a block of N_i of defect l and for which $d_{\mu\nu}^i$ is not

divisible by p. If p^{a} is the maximal order of elements of the defect group D_τ of B_τ, then for all ζ_μ in B_τ the numbers $d^{\zeta}_{\mu\nu}$ belong to the field of the p^{a}th roots of unity. The characters ζ_μ of B_τ belong to the field of the $(p^{a}g')$th roots of unity. If $p\neq 2$ and if B_τ contains y_τ modular characters of G, then at least y_τ of the ordinary characters ζ_μ of B_τ are p-rational. A block of defect d contains at most $p^{d(d+1)/2}$ ordinary characters. For a given p and a given defect d, there exist only a finite number of classes of quadratic forms to which the Cartan form Q of a block of defect d can belong. If the defect of the block B_τ is positive, Q does not represent the number 1. More generally, Q does not represent a form of determinant 1. *C. C. MacDuffee* (Río Piedras, P. R.).

Hirsch, K. A. On infinite soluble groups. III. Proc. London Math. Soc. (2) 49, 184–194 (1946).

In the first two papers of this series [Proc. London Math. Soc. (2) 44, 53–60, 336–344 (1938)] the author initiated the study of S-groups; these are soluble groups, possibly infinite, satisfying the ascending chain condition for subgroups. Thus G is an S-group if there exist normal subgroups $1=G_0<G_1<\cdots<G_i<G_{i+1}<\cdots<G_{n-1}<G_n=G$ such that G_{i+1}/G_i is an Abelian group, generated by a finite number of elements. In the present note the author proves, among other things, for S-groups G, that there exist characteristic subgroups $1=B_0<\cdots<B_j<B_{j+1}<\cdots<B_k=G$ such that B_{j+1}/B_j is nilpotent and contains every nilpotent and normal subgroup of G/B_j; there exists a normal subgroup M of G such that G/M is finite and such that 1 is the only element of finite order in M. Related work may be found in papers of R. Baer [Trans. Amer. Math. Soc. 47, 393–434 (1940); these Rev. 2, 1], H. Fitting [Jber. Deutsch. Math. Verein. 48, 77–141 (1938)] and O. Schmidt [Rec. Math. [Mat. Sbornik] N.S. 8(50), 363–375 (1940); 17(59), 145–162 (1945); these Rev. 2, 214; 7, 511]. Misprint: p. 192, line 20 read "of finite index" instead of "of infinite index."
R. Baer (Urbana, Ill.).

Gelfand, I., and Neumark, M. Unitary representations of the Lorentz group. Acad. Sci. USSR. J. Phys. 10, 93–94 (1946).

This note announces that the authors have determined all the irreducible unitary representations of the homogeneous Lorentz group and of the isomorphic group of unimodular two-dimensional matrices. The representations are (except for the trivial one) all known to be infinite dimensional. The unitary transformations of the representation are given in a form in which they transform functions of two real variables into other such functions and are given for the whole group rather than only for the infinitesimal part of it as was the custom hitherto. It appears that the determination of the representations which is announced is a rigorous one, while most previous work on this question lacked full mathematical rigor. However, it appears that the results corroborate the results which can be obtained by consideration of the infinitesimal operators.
E. P. Wigner (Princeton, N. J.).

Vilenkin, N. On direct decompositions of topological groups. Rec. Math. [Mat. Sbornik] N.S. 19(61), 85–154 (1946). (Russian. English summary)

[An abstract appeared in C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 611–613 (1945); these Rev. 7, 241.] A locally compact group satisfying the second countability axiom and possessing a compact identity-component is designated as of type LKK. If $G_1, G_2, \cdots$ is an enumerable sequence of LKK groups and if in each of these there is distinguished some open compact subgroup H_n, then the author defines a direct sum G of the groups G_n, with respect to the subgroups H_n, by means of the sequences $x=(x_1, x_2, \cdots)$ with $x_n\epsilon G_n$ for all n and $x_n\epsilon H_n$ for almost all n. Appropriately topologized, G is of type LKK. The subgroup H, consisting of those x for which $x_n\epsilon H_n$ for all n, is open and compact in G and is "distinguished" by the decomposition. This leads to a definition of the decomposition of a group G with a distinguished open compact subgroup H into the direct sum of an enumerable set of subgroups of G.

The paper is mainly devoted to the study of zero-dimensional LKK groups of type P; here, for a fixed prime p, $p^n x\to 0$ as $n\to\infty$. It is known that there are four classes of groups which are of type P and of rank one. The principal theorems give conditions on a group of type P that are equivalent to its being decomposable into a direct sum of groups of one of these simple classes. The theorems are further sharpened by the consideration of side conditions limiting the summands to one or another combination of these classes. [It should perhaps be noted here that the author's references to theorems which are consequences of his for the more restricted case of compact zero-dimensional groups overlook the work of H. Ulm, Math. Ann. 107, 774–803 (1933), which contains implicitly a complete classification of these groups.]

A principal, algebraic-topologic, concept pointing up the contrast between locally compact groups on the one hand, and compact or discrete groups on the other, is the notion of a regularly stratified group of type P. To define this, let A denote an arbitrary subset of the group G, ${}_n[A]$ the set of all $x\epsilon A$ such that $p^n x=0$, $p^n A$ the set of all $x\epsilon G$ of the form $p^n y$, where $y\epsilon A$, and let $\bar{A}$ denote the topologic closure of the set A. Then the group G is regularly stratified if, for all k and n, ${}_k\overline{[p^nG]}={}_k\overline{[p^nG]}$ and $p^nG\cap \overline{p^{n+k}G}=p^n(\overline{p^kG})$. It is of some interest that the set on the right side of each of the relations above is automatically included in the other set. The importance of the notion is indicated by the fact that a group of type P is a direct sum of groups of rank one only if it is regularly stratified.

Another important tool is the concept of a regular subgroup H. This is defined by the algebraic condition that $p^nH\cap p^{n+k}G=p^n(H\cap p^kG)$. An early theorem relates the two concepts. Theorem 2 is as follows: if G is of type P, if H is an open compact subgroup, if G with H as distinguished subgroup is the direct sum of regularly stratified subgroups G_n, and if $G_n\cap H$ is regular in G, then G is regularly stratified, and H is regular in G.

Concluding theorems show, among other things, that the group of characters of a group of type P is of type P, and that when one of these is regularly stratified so also is the other. The paper contains a number of examples of groups which are not decomposable into direct sums. The concise six page summary lists nearly all the theorems and the necessary definitions. *L. Zippin* (Flushing, N. Y.).

Segal, I. E. Topological groups in which multiplication of one side is differentiable. Bull. Amer. Math. Soc. 52, 481–487 (1946). [MF 16806]

L'auteur considère une variété G de classe C^1, sur laquelle est définie une structure de groupe topologique telle que, pour tout $a\epsilon G$, la translation à droite, $x\to xa$, soit une application différentiable de G sur elle-même; il faut entendre par là, semble-t-il, que les dérivées partielles des fonctions qui définissent (localement) la translation $x\to xa$ existent et sont

(localement) fonctions continues de (x, a) (et non pas seulement, comme on pourrait croire, fonctions continues de x pour chaque a). Alors, si dx est l'élément de volume invariant à gauche, et si Λ désigne l'ensemble des fonctions (numériques) de classe C^1 sur G, nulles en dehors d'un compact, l'ensemble Γ des fonctions de la forme $h(x)=\int f(xy)g(y)dy$, avec $f\epsilon\Lambda$, $g\epsilon\Lambda$, est contenu dans Λ; de plus, la formule $h(x^{-1})=\int g(xy)f(y)dy$ montre que Λ est invariant par la "symétrie" $x\rightarrow x^{-1}$. L'auteur déduit de là que la symétrie est une application différentiable de G sur lui-même, et qu'il en est par suite de même des translations à gauche, d'où résulte [par exemple, d'après P. A. Smith, Ann. of Math. (2) **44**, 481–513 (1943); ces Rev. **5**, 59] que G est un groupe de Lie. La démonstration contient des complications inutiles, et, semble-t-il, une inexactitude [p. 485: "S and S^{-1} are dense in G, therefore G is covered by neighborhoods where y^{-1} is of class C^1"]; en en conservant l'idée centrale, on peut raisonner comme suit. Soient $\omega_i(x)$ des fonctions formant un système de coordonnées locales en e (élément neutre), et nulles en dehors d'un voisinage convenable V de e; soit $a\epsilon G$; posons $\varphi_i(x)=\int\omega_i(xy)g(y)dy$, avec $g\epsilon\Lambda$, $g\geqq 0$, $g(a)>0$; si g est nul en dehors d'un voisinage suffisamment petit de a, les $\varphi_i(x)$ formeront un système de coordonnées locales au point a^{-1}, comme on le vérifie aisément par différentiation sous le signe $\int$; en posant $x=z^{-1}$, on en conclut, puisque d'après ce qui précède les $\varphi_i(z^{-1})$ sont différentiables sur G, que l'application $z\rightarrow z^{-1}$ est différentiable au point $z=a$. *A. Weil* (São Paulo).

★**Raikov, D. A. Harmonic analysis on commutative groups with the Haar measure and the theory of characters.** Trav. Inst. Math. Stekloff **14**, 86 pp. (1945). (Russian. English summary)

This memoir presents a systematic self-contained development of the character theory for commutative topological groups, with special reference to the harmonic analysis of complex functions defined on such a group, that is, their Fourier-like integral representations in terms of group-characters. Many of the results, or close analogues, are to be found in the literature, particularly in the work of A. Weil [L'intégration dans les groupes topologiques et ses applications, Actualités Sci. Ind., no. 869, Hermann, Paris, 1940, in particular, pp. 94–123, 140–146; these Rev. **3**, 198], I. Gelfand and D. Raikov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 195–198 (1940); these Rev. **2**, 217], I. E. Segal [Proc. Nat. Acad. Sci. U. S. A. **27**, 348–352 (1941); these Rev. **3**, 36], M. Krein [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 484–488 (1941); these Rev. **2**, 316] and D. Raikov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 589–591 (1941); these Rev. **2**, 317]. The presentation is an independent one (the author reports that Weil's book was not received in Moscow until 1945) aiming at a high degree of unification.

The central object of study is a topological group G with a completely additive measure m which is defined on a topologically characterized family of sets and which is (1) right invariant, (2) positive, but not identically $+\infty$, for all "countably open" sets and (3) finite only for subsets of "countably open" sets. A set A is "countably open" (and is necessarily open) if $A=\sum A_n$, where $U_nA_nU_n\subset A$ for a suitable neighborhood U_n of the group-identity. The postulated measure m is assumed to be defined precisely on the Borel field generated by such sets. A measure with these properties is called here a Haar measure. The properties of Haar measure, including its essential uniqueness and the relevant Fubini theorem, are developed without assuming G to be commutative (contrary to what is the case in the remainder of the work). The absolutely integrable complex functions on G constitute a normed ring R with convolution as ring-multiplication. Unless G is discrete, R has no unit. The maximal ideals M in R correspond one-one to the characters χ of G: the homomorphism determined by M is given by $x\rightarrow\int x(g)\chi_M(g)dg$, where χ_M is the associated character. The following propositions are found to be equivalent: (1) R is semi-simple; (2) G has a sufficiency of characters (in the sense that for any g distinct from the group-identity there is some character χ for which $\chi(g)\neq 1$); (3) the "Fourier representation" $\int x(g)\chi(g)dg$ (for functions defined on the family X of all characters χ) has the uniqueness property, vanishing for all χ if and only if x vanishes almost everywhere on G. The characters χ, of course, constitute a commutative group X which can be topologized in a natural manner. Under the topology introduced, X is locally bicompact and $\chi(g)$ is continuous in χ and g together. Haar's fundamental theorem applies to the group X, establishing the existence of a measure μ which can be contracted to one with the properties (1)–(3) above. The author devotes a section to showing how this measure can be derived directly from the relationship of X to G. There he establishes for any bicompact subset Δ of X the formula $\mu(\Delta)=\inf x(e)$, where e is the identity of G and the competition is among those continuous functions x in R such that $\int x(g)\chi(g)dg\geqq 0$ and $\int x(g)\chi(g)dg>1$ for all χ in Δ.

This discussion depends in part upon the theory of "positive definite" functions on G. A complex function φ on G is said to be positive definite if $\sum\varphi(g_kg_l^{-1})\xi_k\bar{\xi}_l\geqq 0$ for all $g_1, \cdots, g_n$ in G and all $\xi_1, \cdots, \xi_n$ in the complex field. For continuous positive definite functions a generalized Bochner representation $\int g(\chi)d\Phi(\chi)$, where $g(\chi)=\chi(g)$ and Φ is a completely additive regular measure on X, is established. From this representation and the observation that the function $\varphi(g)=\int x(gh)\overline{x(h)}dh$ is a continuous positive definite function whenever $|x|^2$ is integrable, a second demonstration of the uniqueness of the Fourier representation easily follows. With the same observation as a basis, it is proved that the generalized Plancherel theorem holds in the form stating that the reciprocal formulas $f(\chi)=\int x(g)\chi(g)dg$, $x(g)=\int f(\chi)\overline{g(\chi)}d\chi$ set up a one-one correspondence between the family of x's with integrable $|x|^2$ on G and the family of f's with integrable $|f|^2$ on X. Finally, it is shown with the aid of the Plancherel theorem that, if G^* is the character group of X, then G^* contains an everywhere dense subgroup isomorphic (topologically and algebraically) to G. Pontrjagin's duality theorem then assumes the form: if G is locally bicompact, G and G^* are isomorphic.

M. H. Stone (Chicago, Ill.).

Dynkin, E. Classification of the simple Lie groups. Rec. Math. [Mat. Sbornik] N.S. **18(60)**, 347–352 (1946). (Russian. English summary)

Partant des résultats de H. Weyl, l'auteur expose une méthode de classification complexe des groupes semi-simples, qui est moins géométrique mais plus rapide que celle de B. L. van der Waerden. Les notions fondamentales sont celles de racine positive (le premier coefficient non nul est positif) et de racine simple (c'est-à-dire racine positive et qui n'est pas une somme de racines positives). Un groupe semi-simple est complètement déterminé par ses racines simples.

Alors l'auteur représente chaque racine simple par un point, en joignant deux points par 0, 1, 2, 3 traits, si l'angle

des vecteurs-racines est respectivement 90°, 120°, 135°, 150°. En se servant de la formule

$$\sum_{i=1}^{p}\sum_{j=1}^{p}(b_i, b_k)=\left(\sum_{i=1}^{p}b_i, \sum_{j=1}^{p}b_j\right)>0$$

(b_i étant des multiples de racines simples), il réussit à exclure un grand nombre de combinaisons et à parvenir aux types bien connus d'É. Cartan. *H. Freudenthal* (Amsterdam).

Climescu, Al. C. Sur les quasicycles. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 1, 5–14 (1946).

A quasicycle A is a finite semigroup generated by one element a. The concept was introduced by A. R. Poole [Amer. J. Math. 59, 23–32 (1937)]. Each quasicycle is characterized by the two integers m and s of the equation $a^{m+1}=a^s$, representing the first duplication among the powers of a. There is exactly one idempotent b in each A, which is the identity of the group contained in A. This group is cyclic and generated by the element ba. The complete structure of an example, $m=9$, $s=8$, is exhibited.

H. H. Campaigne (Arlington, Va.).

Bruck, R. H. Contributions to the theory of loops. Trans. Amer. Math. Soc. 60, 245–354 (1946).

Le terme de loop désigne un ensemble muni d'une opération (non associative) telle que $xa=b$ et $ay=b$ aient toujours une solution et une seule et telle qu'il y ait un élement unité bilatère. Aprés avoir rappelé divers résultats connus l'auteur généralise abstraitement la notion de série centrale [P. Hall, Proc. London Math. Soc. (2) 36, 29–95 (1933)] par celle de π-série.

Puis viennent la π-résolubilité, les ϕ-loops (c'est-à-dire l'ensemble des x tels que $\{x, S\}=G$ entraine $S=G$), l'étude des extensions de la proprieté de Lagrange (l'ordre d'un sous groupe divise l'ordre du groupe). Étude des séries centrales, en relation avec diverses proprietés des groupes d'ordre p^n. Indications sur les séries attachées à l'associateur à gauche, sur l'algèbre non associative attachée à un loop. La seconde partie étudie le cas où chaque élément possède un inverse et en particulier les loops de Moufang (c'est-à-dire tels que $x(y\cdot zy)=(xy\cdot z)y$). Une troisième partie est consacrée à des constructions et à des exemples.

J. Kuntzmann (Grenoble).

Kuntzmann, Jean. Représentations sur un système multiforme. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 21 (1945), 95–99 (1946).

The systems discussed are sets S of elements with one law: an ordered pair ab determines a subset of the system. The system U of all subsets of S has a single-valued multiplication induced in it by that of S. This is similar to an idea introduced by L. W. Griffiths [Amer. J. Math. 60, 345–354 (1938)]. The one-point subsets of S form a special class in U, which generates a subsystem U_S of U. If K is a single-valued system, then a homomorphism of K on U_S is called a "strict representation" of K on S. If the operation in S is single-valued, this reduces to the ordinary notion of representation, since then U_S is isomorphic to S.

Let U_K' be a subsystem of U (built on K) such that each element of K appears in at least one set. This is a generalization of U_K. A homomorphism of U_K' on S will be called a representation of K on S. The usual constructions of multivalued multiplication systems are special cases of this representation, as the hypergroup of double-cosets GaG from a group. A further variation of this mapping is introduced, by which all multivalued systems can be obtained. The reviewer was not able to verify this.

In a multivalued system one can call the mapping $P_a:x\rightarrow ax$ a "permutation." Such permutations generate a single-valued system π (a semigroup). Under some conditions π furnishes a representation. By weakening the concept of representation we get a situation which holds under all conditions. The new concept differs from a representation in the following way. If $AB=CD$ or $AB=C$ in K then $ab=cd$ or $ab=c$ in S, but this correspondence does not carry over to more complicated expressions in these variables. Then π is a representation of S in this weakened sense under all conditions. *H. H. Campaigne* (Arlington, Va.).

NUMBER THEORY

Beeger, N. G. W. H. Note sur la factorisation de quelques grands nombres. Arch. Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. 16, 93–95 (1946).

The author gives a list of nine large primes, thus completing the factorizations of $12^{45}+1$, $11^{35}-1$, $7^{49}+1$, $5^{81}-1$ and a^8+1 for $a=33$, 45, 60, 68 and 120. *D. H. Lehmer.*

Thébault, V. Les récréations mathématiques (parmi les nombres curieux). Mathesis 54, supplement, 119 pp. (1943).

This article contains many problems, with their solutions, on the digital side of the theory of numbers. For example: find all ten digit squares like $44016^2=1937408256$ which have distinct digits. Most of the problems involve the digital properties of squares. Some involve unknown bases of numeration. Pages 62–66 give all 1, 2, 3 and 4 digit square endings for the base 10. Tables like these are useful in recognizing nonsquares. Pages 89–119 are devoted to tables of squares to the base B. Each table extends to the square of B^4. The values of B are 2, 3, 4, 5, 6, 7, 8, 9, 11 and 12.

D. H. Lehmer (Berkeley, Calif.).

***Malengreau, Julien. Contributions à la Théorie des Nombres. II. Étude sur les Nombres de la Forme $B^m\pm 1$. I.** Ganguin & Laubscher, Montreux, 1945. 43 pp.

The author considers the problem of finding the exponent of 2 modulo N when N is of the form $1+2^n+2^m$, $0<n<m$. In a notation different from that of the author, his solution may be put in the following form. Consider a sequence of numbers A_k defined as follows:

$$A_k^2=A_k,\quad A_1=1,\quad A_k=0,\qquad k\leqq 0,$$
$$A_{k+1}+A_{k+1-n}+A_{k+1-m}\equiv\epsilon+A_k+A_{k-n}+A_{k-m}\ (\mathrm{mod}\ 2),$$

where $\epsilon=1$ if $A_k=A_{k-n}=A_{k-m}$, $\epsilon=0$ otherwise. Let r be the least positive integer such that $A_r=A_{r-1}=A_{r-2}=\cdots=A_{r-n+1}\neq A_{r-n}$. Then the exponent of 2 modulo N is either r or $2r$ according as $A_r=0$ or 1. An electro-mechanical device, consisting of relays and contact makers mounted on an endless belt, which could compute the successive A's, is sketched. A photo-electric form of the device is vaguely alluded to. The restriction on the form of N is a serious drawback, but with some complication N may be of one of the forms $\pm 1\pm 2^n\pm 2^m$.

Part II is a brief comment on the author's previous note on divisibility criteria [Mathesis (3) 1, 197–198 (1901)].
D. H. Lehmer (Berkeley, Calif.).

Escott, Edward Brind. Amicable numbers. Scripta Math. 12, 61–72 (1946).

This article contains a list of the 390 pairs of amicable numbers (numbers m, n such that $\sigma(n)=\sigma(m)=m+n$, where $\sigma(k)$ denotes the sum of all the divisors of k) which have been discovered since Pythagoras found 220 and 284. Various methods of discovery are discussed. The twelve other discoverers of amicable numbers are Fermat, Descartes, Euler, Legendre, Paganini, Seelhoff, Dickson, Mason, Poulet, Gerardin, Brown and the author. The latter is responsible for no less than 233 of these pairs. The list is arranged by types of factors. *D. H. Lehmer.*

Yarden, Dov. Table of Fibonacci numbers. Riveon Lematematika 1, 35–37 (1946). (Hebrew)

The two Fibonacci sequences U: 0, 1, 1, 2, 3, 5, 8, $\cdots$; V: 2, 1, 3, 4, 7, 11, 18, $\cdots$, in which each term is the sum of the two preceding terms, are tabulated as far as U_{128} and V_{128} together with their factorizations, when known. The table appears to be an independent calculation. The author is unaware of (a) the previous table of Kraitchik [Recherches sur la Théorie des Nombres, v. 1, Paris, 1924, pp. 77–80] which would have supplied 26 additional complete factorizations, and (b) the Aurifeuillian identity

$$V_{5n}=V_n(V_{2n}-5U_n+3)(V_{2n}+5U_n+3),$$

a powerful tool for factoring V_{5n}. Misprints occur at U_{87}, U_{122}, V_{99} and V_{120}. *D. H. Lehmer* (Berkeley, Calif.).

Ginsburg, Jekuthiel. Iterated exponentials. Scripta Math. 11, 340–353 (1945).

The paper deals with problems attacked by E. T. Bell [Ann. of Math. (2) 35, 258–277 (1934); 39, 539–557 (1938)] and G. T. Williams [Amer. Math. Monthly 52, 323–327 (1945); these Rev. 7, 47] and others. Comparison of results with earlier papers and evaluation of new results is made difficult through an error in § 2. With terms defined as in the paper, the first seven lines of page 341 should read: "where $E_n(x)=e_n(x):e_n(0)=1+\sum_1^\infty A_{nk}x^k/k!$. Thus $E_1(x)=e^x:e^0=e^x$, $E_2(x)=e_2(x)=e_2(x):e_2(0)=e^{e^x}:e=e^{e^x-1}$; $E_3(x)=e_3(x)=e_3(x):e_3(0)=e^{e^{e^x}}:e^e=e^{e(e^{e^x-1}-1)}$, etc. The following relationships can be easily verified:

$$(3)\qquad E_{n+1}(x)=e^{e_n(0)[E_n(x)-1]}=\cdots(?)\cdots,$$

$$(4)\qquad E_{n-1}(x)=1+\frac{1}{e_{n-1}(0)}\cdot\log E_n(x)."$$

Instead, (3) and (4) are given as $E_{n+1}(x)=e^{E_n(x)-1}$ and $E_{n-1}(x)=1+\log E_n(x)$. Since many new theorems are derived directly on the basis of Bell's and Williams' work, a large part of the results may be valid. Several misprints were noted. In the table, page 353, the entry 967 should be replaced by 947; the last column (under $x^7/7!$) must contain errors. *A. J. Kempner* (Boulder, Colo.).

Unger, Georg. Zur Kettenbruchentwicklung von e. Elemente der Math. 1, 93–94 (1946).

The author proves the following theorem. Let B_n be the denominator of the nth convergent of Euler's continued fraction for the Naperian base,

$$e=(2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \cdots).$$

Then B_{3m}, B_{3m-2}, B_{3m-4} and $B_{3m-1}-(-1)^m$ are all divisible by m. These results were noted by D. N. Lehmer from a general theory of such continued fractions [Proc. Nat. Acad. Sci. U. S. A. 4, 214–218 (1918); Amer. J. Math. 40, 375–390 (1918)]. The author contracts the continued fraction and uses methods equivalent to those used by the reviewer [Ann. of Math. (2) 33, 143–150 (1932)]. *D. H. Lehmer.*

Ballieu, Robert. Automorphismes d'un champ de Galois et divisibilité des coefficients polynomiaux par un nombre premier. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 30 (1944), 113–119 (1945).

The author shows that the following two theorems are equivalent. (A) The automorphisms of the Galois field of p^n elements are the transformations $x\to x^{p^k}$. (B) For fixed n, a necessary and sufficient condition that all the binomial coefficients $C_{i,n}$, $1<i<n$, are divisible by a prime p is that n is a power of p. He also gives a simple direct proof of the latter theorem. *G. Whaples* (Madison, Wis.).

Mohr, Ernst. Der grösste gemeinsame Teiler zweier Zahlen bzw. Polynome. Deutsche Math. 7, 593–597 (1944).

Expository article.

Gloden, A. Un nouveau théorème sur les multigrades. Euclides, Madrid 6, no. 64, 377–379 (1946).

In the original, the author's name was spelled Glooden.

Bambah, R. P., and Chowla, S. On integer roots of the unit matrix. Science and Culture 12, 105 (1946).

Let E_{p-1} denote the identity matrix of order $p-1$ and let X be any pth root of E_{p-1} with integral elements, except E_{p-1} itself. It is conjectured that if p is a prime there is a matrix Δ with integral elements such that $X=\Delta^{-1}M\Delta$, where M is made up as follows: $m_{ij}=-1$ if $i=1$, $m_{ij}=1$ if $i=j+1$, $m_{ij}=0$ otherwise. The authors state that they have a proof of this for $p=3$, based on two lemmas which are given.
I. Niven (West Lafayette, Ind.).

Ko, Chao. Some new proofs of Smith's theorem. Acad. Sinica Science Record 1, 308–312 (1945).

Three proofs are given of the theorem: if d is the greatest common divisor of the r-rowed minors of an r by n $(>r)$ matrix with integral elements, the matrix may be augmented with $n-r$ rows of integers to a matrix of determinant d. The third proof is essentially the same as that of A. Bloch [Bull. Soc. Math. France 50, 100–110 (1922)], a possibility acknowledged by the author, to whom Bloch's proof was unavailable. *W. Givens* (Chicago, Ill.).

Todd, J. A. Note on certain reducible polynomials. J. London Math. Soc. 20, 204–209 (1945).

Let m be a positive integer, $M=2^m$, and set $f_m(x)=x^{2M}+ax^M+1$, where a is an integer. It is shown that, if $f_m(x)$ is reducible in the field of rational numbers while $f_{m-1}(x)$ is irreducible, then $m\leqq 2$. The complete discussion of the cases $m=2$ and $m=1$ is given. *R. Brauer* (Toronto, Ont.).

Ljunggren, Wilhelm. Einige Bemerkungen über die Darstellung ganzer Zahlen durch binäre kubische Formen mit positiver Diskriminante. Acta Math. 75, 1–21 (1943). [MF 13204]

Skolem [Skr. Norske Vid. Akad. Oslo. I. 1933, no. 6] obtained a method for solving cubic Diophantine equations with positive discriminant. He considered, in particular, the equation (1) $x^3-3xy^2-y^3=1$, and conjectured that his

method would solve (1) completely. However, he did not carry out the necessary tedious calculations. The author proves that (1) has only the following 6 solutions: (1, 0); (0, −1); (−1, 1); (1, −3); (−3, 2); (2, 1).

Nagell [Norsk Mat. Forenings Skr. (I) no. 2 (1921)] proved that the equation (2) $x^2+x+1=y^n$ has only trivial solutions unless n is a power of 3. For $n=3$ he showed that the solutions of (2) can be obtained if the solutions of (1) are known. It was proved by the reviewer [Bull. Amer. Math. Soc. 49, 712–718 (1943); these Rev. 5, 90] that (2) has no solution for which x is a positive prime number. The author proves that (2) has only two nontrivial solutions, namely $y=7$, $x=18$ or $x=-19$ for $n=3$.

In addition, the author considers the more general equation $x^3-ax^2y-(a+3)xy^2-y^3=1$. *A. Brauer.*

Ljunggren, Wilhelm. On a Diophantine equation. Norske Vid. Selsk. Forh., Trondhjem 18, no. 32, 125–128 (1945).

The author discusses the equation $Cx^2+D=y^n$, where the product of the odd positive integers C, D contains no squared factor greater than 1 and the number of classes of ideals of the field $K(\sqrt{(-CD)})$ is indivisible by the odd positive integer n. For $n>1$ and $D+(-1)^{(D+1)/2}$ exactly divisible by an odd power of 2, the equation is shown to be impossible in integers x, y if $CD\equiv 1 \pmod 4$ or if $CD\equiv 3 \pmod 8$ with $n\not\equiv 0 \pmod 3$. For $n=q>3$, where q is prime, $D+(-1)^{(D+1)/2}=2^{2h}D_1$, $(D_1, 2)=1$, and $CD\not\equiv 7 \pmod 8$, it is also shown to be impossible if $q\not\equiv CD \pmod 8$. *W. H. Gage* (Vancouver, B. C.).

Skolem, Th. Über die ganzen x, für welche ein Polynom $P(u_x, u_{x+1}, \cdots, u_{x+h})=0$ ist, wenn u_x eine gegebene lineare rekurrente Gleichung befriedigt. Avh. Norske Vid. Akad. Oslo. I. 1941, no. 15, 23 pp. (1942). [MF 14159]

In an earlier paper [C. R. Huitième Congrès Math. Scandinaves, Stockholm, 1934, Lund, 1935, pp. 163–188], the author has proved the following theorem. Let $a_0, \cdots, a_n$ be rational integers. The integers $u_0, u_1, \cdots$ may satisfy $\sum_{i=0}^n a_i u_{x+i}=0$, but no other recurrent series with a smaller number of terms. Let $\alpha_1, \cdots, \alpha_n$ be the roots of the corresponding characteristic equation $\sum_{i=0}^n a_i t^i=0$. If no quotient α_i/α_j is a root of unity different from 1, then the equation (1) $\sum_{j=0}^m b_j u_{x+j}=0$ with integral rational coefficients b_j has solutions for an infinite number of values x if and only if the polynomial $\sum_{j=0}^m b_j t^j$ is divisible by $\sum_{i=0}^n a_i t^i$.

In this paper the author gives a more detailed proof of this theorem. At the same time the result is generalized by assuming that the coefficients a_i and b_j are not rational numbers, but numbers of a given algebraic field K. Moreover, the author obtains results, similar to those for the linear form (1), for polynomials $P(u_x, u_{x+1}, \cdots, u_{x+m})$ of higher degree with coefficients from K. *A. Brauer.*

Kuroda, Sigekatu. Über die Pellsche Gleichung. Proc. Imp. Acad. Tokyo 19, 611–612 (1943). [MF 14857]

Let $D>0$ be the discriminant of a quadratic field and let k_0 be the largest odd or even integer not exceeding $\sqrt{D}$, according as $D\equiv 1$ or $0 \pmod 4$. Then

$$\theta=\tfrac{1}{2}(k_0+\sqrt{D})=k_0+\frac{1}{k_1+}\cdots\frac{1}{+k_{n-1}+}\frac{1}{\theta}=\frac{p_n\theta+p_{n-1}}{q_n\theta+q_{n-1}},$$
$$k_\nu=k_{n-\nu}\ (\nu=1, \cdots, n-1).$$

The least positive solution t, u of $t^2-Du^2=\pm 4$ is obtained from the equation $E_0=q_n\theta+q_{n-1}=\frac{1}{2}(t+u\sqrt{D})$. If E_0 has norm $+1$, then $n=2m$. Let

$$\eta=k_m+\frac{1}{k_{m-1}+}\cdots\frac{1}{+k_1+}\frac{1}{\theta}=\frac{p\theta+p'}{q\theta+q'}=\frac{-b+\sqrt{D}}{2a}.$$

Then $a>0$, $\sqrt{E_0}=(q\theta+q')/\sqrt{a}=(q_m\theta+q_{m-1})\sqrt{a}$ is a biquadratic unit, and a is the norm of the primitive ambiguous ideal $[a, a\eta]\neq[1]$ of $R(\sqrt{D})$, which divides E_0+1 or E_0-1 according as m is even or odd. If E_0 has norm -1, then n is odd, and a result pertaining to the field $R(\sqrt{(-1)}, \sqrt{D})$ is obtained. *R. Hull* (Lincoln, Neb.).

Tricomi, Francesco. Su di una formula relativa alla frequenza dei numeri primi. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 77, 120–129 (1942). [MF 16249]

The author sets out results concerning the frequency of prime pairs, triplets, etc., which can be derived by probability methods from the assumption that the distribution of primes can be treated as random. The results so obtained seem to fit fairly closely the facts in the tables and are identical with formulas obtained, by more elaborate methods, by Hardy, Littlewood and Staeckel. Tabulations are made, comparing observed and calculated results, and frequency distribution curves are drawn. *G. Pall.*

Cherwell. Note on the distribution of the intervals between prime numbers. Quart. J. Math., Oxford Ser. 17, 46–62 (1946). [MF 15896]

By simple probability considerations the author deduces heuristically a formula for the frequency of primes in the interval between the squares of the nth and $(n+1)$th primes. From this there follow classical formulas in the asymptotic theory of primes. There is also obtained a functional equation which must be satisfied, approximately, by any function f which represents asymptotically the frequency of primes in the sequence of natural numbers, namely, $f(e^x)=e^{-\phi(x)}$, where $(d/dx)\phi(ax)=e^{-\phi(x)}$. *G. Pall.*

Brauer, Alfred. On the exact number of primes below a given limit. Amer. Math. Monthly 53, 521–523 (1946).

Tietze, Heinrich. Einige Tabellen zur Verteilung der Primzahlen auf Untergruppen der teilerfremden Restklassen nach gegebenem Modul. Abh. Bayer. Akad. Wiss. Math.-Nat. Abt. (N.F.) no. 55, 31 pp. (1944).

The 26 tables of this paper give data (rather meager in all but three cases) on the distribution of primes belonging to certain residue classes modulo m for $m=8, 9, 10, 26, 30$ and 262. In each case these residue classes form a group Γ under multiplication, in fact, a subgroup of the full group H of $\varphi(m)$ residue classes prime to m. Let $\varphi(m)=ih$, where h is the order of Γ, and let $\pi_H(x)$ and $\pi_\Gamma(x)$ denote the number of primes not exceeding x belonging to H and Γ, respectively. Then by the prime number theorem,

$$\pi_H(x)\sim i\pi_\Gamma(x)\sim mx/\varphi(m)\log x$$

as $x\to\infty$. The tables of this paper give values of the difference $\Delta=\pi_H(x)-i\pi_\Gamma(x)$ for x just before and just after each change in the step function $\pi_\Gamma(x)$. The most extensive tables are the last three for $x\leqq 300000$, $m=262$, $i=130$ in which primes of each of the forms $262y+1$, 17 and 259 are considered. The number 262 was chosen because 131 is the least prime having 3, 5, 7, 11, 13 as quadratic residues and 17 and 259 are the extreme primitive roots. The values of Δ show no definite trends except perhaps the case $p=262y+17$ where these primes seem to be more numerous than average. *D. H. Lehmer* (Berkeley, Calif.).

Ziaud Din, M. On formulae in partitions and divisors of a number, derived from symmetric functions. Proc. Nat. Acad. Sci. India. Sect. A. **13**, 221–224 (1943).

The following known formulas are derived:

$$\sigma(n)=p(n-1)+2p(n-2)-5p(n-5)-7p(n-7)+\cdots,$$
$$\sigma(n)=\sigma(n-1)+\sigma(n-2)-\sigma(n-5)-\sigma(n-7)+\cdots,$$
$$np(n)=\sigma(n)+p(1)\sigma(n-1)+p(2)\sigma(n-2)+\cdots +p(n-1)\sigma(1),$$

where $\sigma(k)$ is the sum of the divisors of k, $p(k)$ is the number of unrestricted partitions of k and 0, 1, 2, 5, 7, $\cdots$ are the pentagonal numbers. The last formula is not credited to anyone but must be quite old. The reviewer has not succeeded in locating it in 19th century literature. It is proved without reference by Hellund [Amer. Math. Monthly **42**, 91–93 (1935)]. The methods of the present paper are those of the theory of symmetric functions and determinants, as further developed in the author's previous paper [Math. Student **3**, 141–151 (1935)]. *D. H. Lehmer.*

Kuhn, Pavel. An elementary formula for medium values of Dirichlets divisor problem. Norske Vid. Selsk. Forh., Trondhjem **18**, no. 50, 204–207 (1946).

The author obtains, by elementary methods, an approximate formula with explicit error terms for $\int_0^x f(y)dy$, where $f(y)=\sum_{\nu=1}^{\infty}[y/\nu]$. The integral represents a mean value of $f(y)$ and $f(y)=\sum_{m=1}^{[y]}d(m)$, where $d(m)$ is the number of divisors of m. *R. D. James* (Vancouver, B. C.).

Romanov, N. P. On a special family of infinite unitary matrices. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 295–297 (1946).

The author restates the main properties of the family of infinite unitary matrices which he derived in an earlier paper [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 3–34 (1946); these Rev. **8**, 9] in terms of an arbitrary sequence of differentiable functions satisfying certain conditions. Orthogonality relations and other properties are deduced from the differential equations satisfied by the matrices. *R. A. Rankin* (Cambridge, England).

Chagleev, P. On a certain orthonormalized sequence. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 271–276 (1946). (Russian. English summary)

The author shows that the system

$$\rho_n(x)=\{\varphi_2(n)\}^{-\frac{1}{2}}\sum_{d|n}\mu(n/d)dF(dx),\qquad n\text{ odd},$$

is orthonormal in $L^2(0, 1)$. Here $F(x)=1$ $(0\leqq x<\frac{1}{2})$, $F(x)=-1$ $(\frac{1}{2}<x\leqq 1)$, $\mu(n)$ is the Möbius function and $\varphi_2(n)=\sum_{d|n}\mu(n/d)d^2$. The system is shown to be complete for the class of functions $\lambda(x)$ (of period 1) belonging to $L^2(0, 1)$ for which $\lambda(1-x)=-\lambda(x)$ and $\lambda(\frac{1}{2}-x)=\lambda(x)$. The method employed is similar to one used by N. P. Romanoff [Rec. Math. [Mat. Sbornik] N.S. **16**(58), 353–364 (1946); these Rev. **7**, 365] whose orthonormal system $\{\psi_n(x)\}$ is connected with $\{\rho_n(x)\}$ by a simple relation. There are several misprints. *R. A. Rankin* (Cambridge, England).

Chatrowsky, L. Sur le théorème de Erdös-Raikov. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 301–310 (1945). (Russian. · French summary) [MF 15360]

This article deals with a generalization to n-dimensional point sets of a theorem of Raikov for the case $n=1$, on the density of the sum of two point sets [Rec. Math. [Mat. Sbornik] N.S. **5**(47), 425–440 (1939); these Rev. **1**, 296]; Raikov's theorem resembles that of Erdös on the arithmetic or asymptotic densities of the sum of sequences of integers [Acta Arithmet. **1**, 197–200 (1936); Trav. Inst. Math. Tbilissi **3**, 217–223 (1938)]. If A and B are two sets of points $\bar{a}=(a_1, \cdots, a_n)$ and $\bar{b}=(b_1, \cdots, b_n)$, with nonnegative coordinates a_i, b_i, the sum $C=A+B$ is defined to consist of all points $\bar{a}$, $\bar{b}$, or $\bar{a}+\bar{b}=(a_1+b_1, \cdots, a_n+b_n)$; $X(\bar{m})$ denotes the inner measure of the part of the set X satisfying $\bar{x}\subset\bar{m}$ (i.e., satisfying $x_i\leqq m_i$, $i=1, \cdots, n$). A subset B of M is called a weak basis if the limit (when the number of summands tends to infinity) of $B+B+\cdots+B$ is everywhere dense in M; a weak asymptotic basis if this is true of the subset of M satisfying $\bar{m}\supset\bar{m}^{(0)}$ for some point $\bar{m}^{(0)}$. The lower bound of $A(\bar{m})/M(\bar{m})$, for all $\bar{m}$, is called the density of A in M; the lower limit is called the asymptotic density. The height $h(\bar{x})$ of a weak basis B at a point $\bar{x}$ is the lower limit as $\bar{y}\to\bar{x}$ of $g(\bar{y})$, where $g(\bar{y})$ is the least integer n for which $\bar{y}$ is in nB; the height λ of B is the upper bound for all $\bar{m}$ in M of $I/M(\bar{m})$, where $I=\int h(\bar{x})d\bar{x}$ over $\bar{x}\subset\bar{m}$. The asymptotic height λ^* of B is the upper limit, for all $\bar{m}^{(0)}$ in M, of the heights of B over the subsets $\bar{m}\supset\bar{m}^{(0)}$ of M. (1) If B is a weak basis of height λ_0 in the region $\bar{m}\subset\bar{m}^{(0)}$, then for any subset A of M of density $\alpha_0<2^{1-n}$ in this region, the density γ_0 of $A+B$ satisfies

$$\gamma_0\geqq\alpha_0\{1+2^{-n}\lambda_0^{-1}(1-2^{n-1}\alpha_0)\}.$$

If $\bar{m}^{(0)}$ is left free, this holds for the densities in M. (2) If B is a weak asymptotic basis of asy.nptotic height λ^*, then for any subset A of M of asymptotic density $\alpha^*<2^{1-n}$, the asymptotic density γ^* of $A+B$ satisfies

$$\gamma^*\geqq\alpha^*\{1+2^{-n}\lambda^{*-1}(1-2^{n-1}\alpha^*)\}.$$

G. Pall (Chicago, Ill.).

Mahler, Kurt. A problem of Diophantine approximation in quaternions. Proc. London Math. Soc. (2) **48**, 435–466 (1945).

In the first chapter several inequalities in the field of quaternions are derived. Chapter II deals with Hermitian forms in quaternions and their reduction. By combining the results of both chapters the author proves his main theorem which forms an equivalent to a well-known theorem of Minkowski on the product of linear (inhomogeneous) forms. If α, β, γ, δ, ρ, σ are constant quaternions such that $\alpha\bar{\alpha}\delta\bar{\delta}+\beta\bar{\beta}\gamma\bar{\gamma}-\alpha\bar{\gamma}\delta\bar{\beta}-\beta\bar{\delta}\gamma\bar{\alpha}=1$, then there are two integral quaternions x, y satisfying $|\alpha x+\beta y+\rho|\,|\gamma x+\delta y+\sigma|\leqq\frac{1}{4}$. *J. F. Koksma* (Amsterdam).

Chaundy, T. W. The arithmetic minima of positive quadratic forms. I. Quart. J. Math., Oxford Ser. **17**, 166–192 (1946).

Let F be a positive definite quadratic form in n variables with given coefficients a_{rs}. If the variables x_r are restricted to integral values, not all zero, the corresponding values of F are necessarily positive, and will somewhere attain a minimum. Multiplying F by a suitable number, we can arrange that this minimum shall be 2. Then the determinant $\Delta=|a_{rs}|$ of F is a function of the coefficients. If we vary the coefficients, Δ will have an absolute minimum value Δ_n, depending only on n; e.g., $\Delta_1=2$ and $\Delta_2=3$, these being the determinants of the forms $2x_1^2$ and $2(x_1^2+x_1x_2+x_2^2)$. Using an inductive method, considerably simpler than that used by Gauss, Korkine and Zolotareff, Hofreiter, and Blichfeldt [Math. Z. **39**, 1–15 (1934)], he corroborates the known values $\Delta_3=\Delta_4=\Delta_5=4$, $\Delta_6=3$, $\Delta_7=2$, $\Delta_8=1$, and adds the

new values $\Delta_9=1$, $\Delta_{10}=\frac{3}{4}$. The nonary form giving $\Delta_9=1$ is equivalent to the extreme form T_9 of Korkine and Zolotareff [Math. Ann. 6, 366–389 (1873)], but the denary form giving $\Delta_{10}=\frac{3}{4}$ is entirely new.

Some readers have doubted the rigor of the author's argument because in varying the coefficients (as in the change from G_2 to G_2' on page 169) he admits forms which might be expected to attain values less than 2. The author agrees that this is a valid objection, but he believes the same results can be obtained by an improved method which he is now developing. *H. S. M. Coxeter* (Notre Dame, Ind.).

***Brandt, H. Über quadratische Kern- und Stammformen.** Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser, 87–104, Füssli, Zürich, 1945.

A Kernform is an integral quadratic form of nonzero determinant which cannot be derived from any other such form by an integral linear transformation of determinant greater than 1. If also no integer multiple of the form can be so derived it is called a Stammform. Necessary and sufficient conditions for a form to be either a Kernform or Stammform are obtained. One can confine attention to transformations of a prime determinant p, where p^2 divides the discriminant (for a Kernform) or p divides the discriminant (for a Stammform). A form f is not a Kernform with respect to a prime p, if and only if the congruence $f(x_1, \cdots, x_n)\equiv 0 \pmod{p^2}$ is solvable primitively with the n linear forms $\partial f/\partial x_i$ all divisible by p. Simple conditions in terms of the generic characters and the expressions of f in canonical forms mod p^2 or mod 8 are obtained. This investigation has useful applications. It arises from the study of families of forms suggested by the theory of quadratic forms in the field of rationals and was inspired by a letter to the author from A. Speiser. [See Brandt, Jber. Deutsch. Math.-Verein. 47, Abt. 1, 149–159 (1937).] *G. Pall.*

Hua, Loo-keng, and Shih, Wei-Tong. On the Euclidean algorithm in the real quadratic fields. Acad. Sinica Science Record 1, 319 (1945).

This is an abstract of the authors' paper in Amer. J. Math. 67, 209–211 (1945); these Rev. 6, 256. *A. Brauer* (Chapel Hill, N. C.).

Rédei, Ladislaus. Über einige merkwürdige Polynome in endlichen Körpern mit zahlentheoretischen Beziehungen. Acta Univ. Szeged. Sect. Sci. Math. 11, 39–54 (1946).

Let Q denote the $GF(p^n)$, $p\neq 2$. The author first remarks that a polynomial in Q, all of whose values are squares in Q, is not necessarily a square [see Dickson, Trans. Amer. Math. Soc. 10, 109–122 (1909)]. Next he defines

$$\Phi_{++}(x)=\tfrac{1}{2}\{1+x^{\frac{1}{2}(q+1)}+(1-x)^{\frac{1}{2}(q-1)}\},$$
$$\Phi_{+-}(x)=2x^{-1}\{1+x^{\frac{1}{2}(q+1)}-(1-x)^{\frac{1}{2}(q+1)}\}$$

and two similar polynomials $\Phi_{-+}(x)$, $\Phi_{--}(x)$; also four polynomials

$$\varphi_{\rho\sigma}(x)=(1+\rho x^{\frac{1}{2}(q-1)},\quad 1+\sigma(1-x)^{\frac{1}{2}(q-1)}),\qquad \rho,\sigma=\pm 1.$$

The values of $\Phi_{\rho\sigma}(x)$ are all squares in Q; indeed,

$$\Phi_{\rho\sigma}(x)=\varphi^2_{\rho\sigma}(x).$$

Other typical results are

$$1+x^{\frac{1}{2}(q-1)}=\varphi_{++}(x)\varphi_{+-}(x),\quad 1-x^{\frac{1}{2}(q-1)}=\varphi_{-+}(x)\varphi_{--}(x),$$
$$\varphi_{++}(x)=A(x)+xB(x),\quad \varphi_{+-}(x)=2\{A(x)+B(x)\},$$
$$\varphi_{-+}(x)=A(x),\quad \varphi_{--}(x)=-2B(x),$$

where $\{(1+x)(1+x^p)\cdots(1+x^{p^{n-1}})\}^{\frac{1}{2}(p-1)}=A(x^2)+xB(x^2)$.
L. Carlitz (Durham, N. C.).

Rédei, Ladislaus. Zur Theorie der Gleichungen in endlichen Körpern. Acta Univ. Szeged. Sect. Sci. Math. 11, 63–70 (1946).

Let k be the finite field $GF[q]$ of $q=p^m$ elements, p prime, and let $F=F(x_1, \cdots, x_n)$ be an arbitrary nonconstant polynomial of total degree g in $x_1, \cdots, x_n$, with coefficients in k. Since $x^q=x$ for every x in k, it is assumed, without loss of generality regarding the solvability in k of $F=0$, that the degree of F in each x_i is at most $q-1$. The rank of F is the least integer r for which there exists a nonsingular linear homogeneous transformation, in k, carrying F into a polynomial involving exactly r indeterminates. The author obtains the criterion: $F=0$ has no solutions if and only if $F^{q-1}-1$ is a linear combination $H_i(x)(x_i^q-x_i)$, with polynomial coefficients $H_i(x)$ in $k[x_1, \cdots, x_n]$. One may assume, furthermore, that $H_i(x)$, $i=1, \cdots, n-1$, is of degree at most $q-1$ in $x_{i+1}, \cdots, x_n$, and then the maximum of the degrees of $H_i(x)$ is $gq-g-q$. From this criterion follows the theorem that $F=0$ has solutions if $\bar{F}^{q-1}$ has at least one term whose degree in each x_i is at most $q-1$, where $\bar{F}$ is the sum of all terms of F of the maximum degree g. As an example, if $g\mid(p-1)$, and p does not divide $a_1\cdots a_g$, then the rational integral congruence $a_1x_1^g+\cdots+a_gx_g^g\equiv c \pmod{p}$ has solutions for every integer c. The condition of the theorem requires $g\leqq n$, and, indeed, $g\leqq \bar{r}$, the rank of $\bar{F}$. The author's results for the equation $F=0$, and further results for the special case $F-1=0$, F homogeneous, support his unproved conjecture: if k is a prime field ($m=1$) and $g\leqq r$, then $F=0$ has solutions. Examples due to Warning [Abh. Math. Sem. Hamburgischen Univ. 11, 76–83 (1935)] disprove that $g\leqq r$ will be sufficient, in general, if $m>1$, and show that $g\leqq r$ is necessary, in general, when $m=1$.
R. Hull (Lincoln, Neb.).

Rédei, Ladislaus. Über die Gleichungen dritten und vierten Grades in endlichen Körpern. Acta Univ. Szeged. Sect. Sci. Math. 11, 96–105 (1946).

The paper is concerned with the possibility of solving by "radicals" equations of degree three and four with coefficients in a finite field $GF(p^t)$ without enlarging the ground field ($p\neq 2, 3$). As the author points out, a complete solution is not obtained. The theorems are too complicated to quote here. At the end of the paper the author determines the number of cubic and biquadratic equations having a given number of roots in $GF(p^t)$ and incidentally corrects some results of Skolem [Norske Vid. Selsk. Forh., Trondhjem 14, 161–164 (1942)] on cubic congruences. *L. Carlitz.*

Rédei, Ladislaus. Über eindeutig umkehrbare Polynome in endlichen Körpern. Acta Univ. Szeged. Sect. Sci. Math. 11, 85–92 (1946).

The main result of the paper is contained in the following theorem. Let k denote the $GF(q)$, $q=p^t$, $p>2$; suppose that $(n, q+1)=1$ and let α denote a nonsquare in k. Put $(x+\alpha^{\frac{1}{2}})^n=g(x)+h(x)\alpha^{\frac{1}{2}}$, where $\alpha^{\frac{1}{2}}\epsilon GF(q^2)$ and $g(x)$, $h(x)\epsilon k[x]$. Then the rational function $f_n(x)=g(x)/h(x)$ represents a permutation on $0, 1, \cdots, p-1$. If P_n denotes the permutation $x\rightarrow f_n(x)$, its order is the smallest positive integer r such that $n^r\equiv 1 \pmod{(q+1)}$. The P_n constitute an Abelian group of order $\varphi(q+1)$; in particular, $f_m(f_n(x))=f_{mn}(x)$. From this it follows that for n odd one can always construct rational functions of degree n that represent permutations on $0, 1, \cdots, p-1$. *L. Carlitz* (Durham, N. C.).

Rédei, Ladislaus. Bemerkung zu einer Arbeit von R. Fueter über die Klassenkörpertheorie. Acta Univ. Szeged. Sect. Sci. Math. **11**, 37–38 (1946).

L'auteur fait remarquer que le résultat suivant de Fueter [Jber. Deutsch. Math. Verein. **20**, 1–47 (1911), p. 46]: si $D\equiv 1 \pmod{8}$, la parité du nombre des classes d'idéaux contenues dans un genre du corps $k(\sqrt{(-D)})$ (où k est le corps rationnel) est une condition nécéssaire pour que la norme de l'unité fondamentale de $k(\sqrt{D})$ soit -1, est tautologique, car il résulte d'un travail de l'auteur et de Reichardt [J. Reine Angew. Math. **170**, 69–74 (1933)] que, si $D\equiv 1 \pmod{8}$, les genres du corps $k(\sqrt{(-D)})$ contiennent toujours un nombre pair de classes d'idéaux. Ensuite, l'auteur indique qu'il existe des relations véritables entre les groupes de classes d'idéaux de $k(\sqrt{(-D)})$ et de $k(\sqrt{D})$, tels le résultat du travail précédent de l'auteur que la divisibilité par 4 des invariants pairs du premier groupe entraine celle des invariants pairs de la seconde, et le résultat analogue pour leur divisibilité par 8 que donne un autre travail de l'auteur [J. Reine Angew. Math. **180**, 1–43 (1938)], ainsi que les reciproques partiels de ces résultats. L'auteur éspère que son travail récent [J. Reine Angew. Math. **186**, 80–90 (1944); ces Rev. **7**, 111] permettra d'obtenir des relations plus générales entre les 2-groupes de Sylow de ces groupes, et, plus généralement, entre les p-groupes de Sylow des groupes des classes d'idéaux de couples appropriés de corps quadratiques, et cite un résultat de Scholz [J. Reine Angew. Math. **166**, 201–203 (1932)] dans cette direction, où $p=3$ et où les corps sont $k(\sqrt{d})$ et $k(\sqrt{(-3d)})$.

M. Krasner (Paris).

ANALYSIS

Calculus

Obrechkoff, Nikola. Sur les différences divisées et la formule de Lagrange. C. R. Acad. Sci. Paris **223**, 370–372 (1946).

Let $f(x)$ be n times differentiable for $x>a$ and let there be an integer m $(0\leqq m<n)$ such that along a sequence $y_1, y_2, \cdots$, with $y_\nu\to+\infty$, we have $\lim f(y_\nu)y_\nu^{-m}=A$ as $\nu\to\infty$. Moreover, let the integral $\int_0^\infty u^{n-m-1}|f^{(n)}(u+x)|du$ converge for $x>a$. Under these assumptions it is shown that

$$f(x_0, x_1, \cdots, x_n)=A+\frac{(-1)^{n-m}}{(n-m-1)!}\int_0^{t_1}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{m-1}}\varphi(\omega)dt_m,$$

where

$$\varphi(x)=\int_0^\infty u^{n-m-1}f^{(n)}(u+x)du,$$

$$\omega=x_0(1-t_1)+x_1(t_1-t_2)+\cdots+x_{m-1}(t_{m-1}-t_m)+x_mt_m.$$

I. J. Schoenberg (Philadelphia, Pa.).

Brelot, M. Sur la formule de Taylor. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 91–93 (1946).

Taylor's theorem with remainder is proved on the basis of Cauchy's mean value theorem. *P. Franklin.*

Ott, H. Die Sattelpunktsmethode in der Umgebung eines Pols mit Anwendungen auf die Wellenoptik und Akustik. Ann. Physik (5) **43**, 393–403 (1943).

The author considers the integral

$$J(x)=\int_C A(\theta)\exp(ix\cos(\theta-\alpha))d\theta,$$

where C is a contour in the complex θ-plane and goes from $\theta=-\frac{1}{2}\pi+i\infty$ through $\theta=0$ to $\theta=\frac{1}{2}\pi-i\infty$, x is a real variable, α is a real number between 0 and $\frac{1}{2}\pi$ and $A(\theta)$ is a regular analytic function in the strip $-\frac{1}{2}\pi<\Re\theta<\frac{1}{2}\pi$ except for a pole of the first order at some point $\theta=\theta_p$. The problem is to obtain the asymptotic behavior of J for large and small values of x. The method is formal and is a slight modification of that used by Pauli [Physical Rev. (2) **54**, 924–931 (1938)] in an investigation of a particular integral.

P. Hartman (Baltimore, Md.).

Géhéniau, J. Sur une propriété de la dérivée par rapport à t d'une intégrale p-uple. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 144–150 (1945).

In a space of n dimensions $x_1, \cdots, x_n$, consider a family of varieties $V_p(t)$ of p dimensions defined by $S_r(x_1, \cdots, x_n; t)=0$, $r=1, \cdots, n-p$. Here t is the parameter of the family. Select the functions $X_1(x_1, \cdots, x_n; t), \cdots, X_n$ so that the S_r are the invariations relative to the trajectories $dx^a/X_a=dt$; then $\sum_a X_a\partial S_r/\partial x^a+\partial S_r/\partial t=0$ on $S^r=0$. Consider a symbolic form $A_p=\sum F_{i_1\cdots i_p}dx_{i_1}\cdots dx_{i_p}$ defined on $V_p(t)$, where the $F_{i_1\cdots i_p}$ are functions of $x_1, \cdots, x_n, t$, and form the integral over a closed portion R of $V_p(t)$, $I(t)=\int_R A_p$. If BR denotes the boundary of R then the known formula for the derivative $I(t)$ is of the form

$$\frac{dI}{dt}=\int_R\frac{\partial A_p}{\partial t}+\int_R H_p+\int_{BR}H_{p-1},$$

$$\frac{\partial A_p}{\partial t}=\frac{\partial F_{i_1\cdots i_p}}{\partial t}dx_{i_1}\cdots dx_{i_p},$$

where H_p and H_{p-1} are symbolic forms involving the functions $X_1, \cdots, X_n$. If $S^*(x_1, \cdots, x_n; t)$ is the additional function needed in defining the boundary of R, the author shows that the explicit appearance of the $X_1, \cdots, X_n$ in H_p and H_{p-1} can be eliminated by use of Jacobians of the functions $S_1, \cdots, S_n, S^*$. *F. G. Dressel* (Durham, N. C.).

Theory of Sets, Theory of Functions of Real Variables

Hewitt, Edwin. A remark on density characters. Bull. Amer. Math. Soc. **52**, 641–643 (1946).

Soit X un espace topologique, son type de densité est par définition la puissance minimale $\Xi(X)$ d'un sousensemble de X dense dans X. D'aprés Pospíšil [Časopis Pěst. Mat. Fys. **67**, 89–96 (1938); Ann. of Math. (2) **38**, 845–846 (1937)], si X est un espace de Hausdorff, la puissance $|X|$ de X et $\Xi(X)$ sont liés par l'inégalité $|X|\leqq 2^{2^{\Xi(X)}}$. La signe de l'égalité vaut pour certains espaces aux voisinages particulièrement riches, par exemple pour les prolongements compacts d'espaces discrets obtenus par la construction de Stone-Čech.

L'auteur prouve que l'espace P, produit topologique cartésien de $2^{\mathfrak{m}}$ espaces de Hausdorff H_λ ($\mathfrak{m}$ nombre cardinal infini) dont chacun a une puissance non inférieure à 2 et non supérieure à $\mathfrak{m}$, en est un nouvel exemple. Corollaires: l'espace des fonctions numériques d'une variable réelle muni de la topologie du produit cartésien contient un sousensemble dénombrable dense; l'espace $\{0, 1\}^c$, $c=2^{\aleph_0}$, contient un sous-ensemble dénombrable dense. *C. Pauc.*

Kudryavtzev, L. D., and Rodnyanski, A. M. On the power of the system of components of sets of the type F_σ. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 3–5 (1946).

The authors state the following theorem. Let R be any set which is an F_σ. Then the system of components of R is finite, denumerable or of the power of the continuum. Their proof assumes that the set R is a subset of a space X which is a countable union of bicompact spaces and that X satisfies the second axiom of countability. Under these restrictions, the theorem is true. *E. Hewitt* (Bryn Mawr, Pa.).

Sierpiński, W. Sur une proposition de Mlle. S. Piccard. Comment. Math. Helv. **18**, 349–352 (1946).

S. Piccard showed [Sur les ensembles de distances des ensembles de points d'un espace euclidien, Mém. Univ. Neuchâtel **13**, 1939, pp. 68–71; these Rev. **2**, 129], by means of the continuum hypothesis, that there exists an L-nonmeasurable linear set E which is congruent to its complement, does not possess the property of Baire and is such that all positive numbers, with the exception of a denumerable set, are realized as distances between two points of E. The present note proves this result without resort to the continuum hypothesis. *H. Blumberg* (Columbus, Ohio).

Sierpiński, Wacław. Sur la non-invariance topologique de la propriété λ'. Fund. Math. **33**, 264–268 (1945).

L'auteur étudie les relations entre diverses propriétés des ensembles linéaires: propriété λ de C. Kuratowski [Fund. Math. **21**, 127–128 (1933)], propriété λ' définie par l'auteur [C. R. Soc. Sci. Lett. Varsovie **30**, 257–259 (1937)], et ensemble concentré de A. S. Besicovitch [Acta Math. **62**, 289–300 (1934)]. Un ensemble E est dit jouir de la propriété λ si pour tout dénombrable D contenu dans E il existe un G_δ contenant D et dont l'intersection avec E se réduit à D. La propriété λ' se definit en supprimant dans l'énoncé de λ "contenu dans E." Un ensemble E est dit concentré s'il existe un dénombrable D tel que tout ouvert contenant D contient E à un dénombrable près.

L'auteur établit d'abord que λ' équivaut à la condition de ne contenir aucune partie non dénombrable concentrée. Il démontre ensuite que, contrairement à λ, la propriété λ' ne se conserve pas par une homéomorphie sur E. Il utilise pour cela le résultat suivant de F. Rothberger, énoncé sans référence. Il existe un ensemble concentré H formé de nombres irrationnels qu'on peut appliquer d'une manière biunivoque et continue sur un intervalle fermé. Les démonstrations font appel à l'hypothèse du continu. En terminant, l'auteur introduit la propriété λ' relative au plan et la relie à la précédente. *R. de Possel* (Alger).

Sierpiński, Wacław. Sur deux conséquences d'un théorème de Hausdorff. Fund. Math. **33**, 269–272 (1945). [MF 16877]

Without the hypothesis of the continuum, F. Hausdorff demonstrated that the line is a sum of a transfinite series of type Ω of actually increasing G_δ sets. Using this theorem, Sierpiński shows that (1) there is a linear set E such that D linear and denumerable implies D is a G_δ relative to $E+D$; (2) there is a decomposition of the line into nonempty disjoint $F_{\sigma\delta}$ sets. Both results were previously known under the hypothesis of the continuum. There is a discussion of the relation of these results to others and it is pointed out that it is still not known whether, without the hypothesis of the continuum, $F_{\sigma\delta}$ can be replaced in statement (2) by G_δ. *J. F. Randolph* (Oberlin, Ohio).

Sierpiński, Wacław. Sur le paradoxe de la sphère. Fund. Math. **33**, 235–244 (1945). [MF 16876]

Let us say that 2 subsets A and B of Euclidean 3-space E_3 are n-equivalent by decomposition if they are respectively expressible as the sum of n (mutually) disjoint subsets A_i, B_i, $i=1, \cdots, n$: $A=\sum_{i=1}^n A_i$, $B=\sum_{i=1}^n B_i$ (i.e., $A_iA_j=B_iB_j=0$ for all $i\neq j$), such that A_i is congruent to B_i, $i=1, \cdots, n$. The author proves the theorem: every sphere S of E_3 is the sum of $2^{\aleph_0}$ disjoint subsets of which each is n-equivalent to S with $n=9$. The proof utilizes a result of Hausdorff and an idea of J. von Neumann and connects with less assertive results of Hausdorff, Banach and Tarski, von Neumann, and the author. *H. Blumberg* (Columbus, Ohio).

Frenkel, Yanny. Simplified demonstration of a theorem of Lebesgue. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 133–135 (1946). (Spanish) [MF 16916]

A simple proof is given, for sets on the real axis, for the theorem stating that the points of a set which are not points of outer density form a set of measure zero. *J. V. Wehausen* (Falls Church, Va.).

Amerio, Luigi. Sulle famiglie di insiemi. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 699–702 (1941).

Let C be a family of point sets in the (xy)-plane; then $\bar{I}$ is called a set of accumulation of C if for every $\rho>0$ there exists at least one set J of C contained in a ρ-neighborhood of $\bar{I}$ and such that for each point P of $\bar{I}$ there is at least one point Q of J with $\overline{PQ}<\rho$. The author proves the following generalization of the Bolzano-Weierstrass theorem. If C is a family of closed sets, all contained in a square Q, then in Q there exists at least one closed set of accumulation of C. For the proof of this theorem essential use is made of the Ascoli theorem about the existence of a function of accumulation for a family of equicontinuous and uniformly bounded functions. The author then applies his result to obtain the following generalization of the Ascoli theorem. If K is a family of equicontinuous and uniformly bounded functions $f(x, y)$, each of which is defined on a closed set contained in Q, then there exists at least one function $\varphi(x, y)$, defined and continuous on a closed set in Q, which is a function of accumulation for K. It is obvious that all these results can be generalized to n dimensions. *A. Rosenthal.*

Korevaar, J. A theorem on uniform convergence. Nederl. Akad. Wetensch., Proc. **49**, 752–757 (1946).

The following theorem is proved and applied to establish the completeness of several sets of functions. If $\{f_n(x)\}$ is a sequence of continuous uniformly bounded functions on (0, 1) which converges to a continuous limit $f(x)$ uniformly in every interval $0\leqq x\leqq\lambda$, $0<\lambda<1$, then a subsequence exists which is $(C, 1)$-summable to $f(x)$, uniformly on (0, 1). *H. Pollard* (Ithaca, N. Y.).

Pospíšil, Bedřich. Eine Bemerkung über Funktionenfolgen. Časopis Pěst. Mat. Fys. **70**, 119–121 (1941). (German. Czech summary)

It is shown, by the use of topological methods from the theory of Boolean algebras, that for any sequence of Lebesgue (or Borel) measurable functions g_n there is a like function g of which each g_n is a continuous function: $g_n=f_n(g)$, where f_n is continuous. Generalizations and refinements are given. *M. H. Stone* (Chicago, Ill.).

Vicente Gonçalves, J. Sur la primitive des différentielles totales. Revista Fac. Ci. Univ. Coimbra **9**, 65–68 (1941).

Zahorski, Zygmunt. Sur les dérivées des fonctions partout dérivables. C. R. Acad. Sci. Paris **223**, 415–417 (1946).

The author states many theorems on Baire functions of the first class satisfying the Darboux condition and, in particular, on derivatives of everywhere differentiable functions. No proofs are indicated. *A. Rosenthal.*

Zahorski, Zygmunt. Problèmes de la théorie des ensembles et des fonctions. C. R. Acad. Sci. Paris **223**, 449–451 (1946).

Many theorems are stated without proofs. Thus a more extensive review has to be postponed until a detailed publication appears. The author discusses (I) functions with infinitely many derivatives and relations to their Taylor expansions; (II) quasi-analytic functions in a generalized sense; (III) parametric representations $x_i=f_i(t)$ of curves such that the derivatives $f_i'(t)$ exist. Under (I) the author states that the proof given by A. Pringsheim [Math. Ann. **42**, 153–184 (1893), in particular, p. 180] for his theorem, "if for every $x\epsilon[a, b]$ the radius of convergence of the Taylor series of $f(x)$ is greater than $\delta>0$, then $f(x)$ is regular in $[a, b]$" is not correct. This has already been observed; correct proofs were given by Boas [Bull. Amer. Math. Soc. **41**, 233–236 (1935)] and Ganapathy Iyer [C. R. Acad. Sci. Paris **199**, 1371–1373 (1934)]. *A. Rosenthal.*

Raimondi, Elba. On continuous nondifferentiable functions. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 247–253 (1946). (Spanish) [MF 16928]

It has been proved by G. H. Hardy [Trans. Amer. Math. Soc. **17**, 301–325 (1916)] that the function

$$f(x)=\sum_{n=0}^{\infty}c^n\cos\pi a^n x$$

has no finite derivative at any point if $ac\geqq 1$. The author investigates graphically the shape of such functions for a value of $ac=1$ $(a=5,\ c=\frac{1}{5})$ and $ac<1$ $(a=2,\ c=\frac{1}{3})$. The graphs are said to be accurately drawn to within the width of the printed line. As is pointed out, one cannot tell from looking at the graphs that the two functions differ so radically with regard to the property of being differentiable.

J. V. Wehausen (Falls Church, Va.).

Shukla, P. D. On the differentiability of functions. Proc. Benares Math. Soc. (N.S.) **7**, 27–34 (1945).

Necessary conditions for the existence of a finite differential coefficient at a point are given in terms of sequential approach to the point. Similar results for monotonic functions had previously been established by the author [Bull. Calcutta Math. Soc. **37**, 9–14 (1945); these Rev. **7**, 9].

P. Civin (Eugene, Ore.).

Martinez Salas, J. On a note concerning the Lebesgue set. Revista Mat. Hisp.-Amer. (4) **6**, 127–131 (1946). (Spanish)

The author shows that for a bounded integrable function the points x of the Lebesgue set are those for which, for every positive ϵ, the set of points t such that $|f(t)-f(x)|<\epsilon$ has density 1 on the right and on the left. For an unbounded function, the same condition applies after the exclusion of a set of arbitrarily small positive measure.

R. P. Boas, Jr. (Providence, R. I.).

Faedo, Sandro. Su una proposizione fondamentale per le funzioni d'intervallo. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **12**, 593–614 (1942).

L. Tonelli [Ann. Scuola Norm. Super. Pisa (2) **8**, 309–321 (1939); these Rev. **1**, 303] called an interval function $\Phi(\delta)$, defined for the subintervals δ of an interval (a, b), "approximately subadditive" if to each $\epsilon>0$ there is a $\lambda_\epsilon>0$, such that for every interval δ with $|\delta|\leqq\lambda_\epsilon$ and for every subdivision of δ into intervals $\delta_1, \cdots, \delta_n$,

$$\Phi(\delta)\leqq\Phi(\delta_1)+\cdots+\Phi(\delta_n)+\epsilon|\delta|,$$

where $|\delta|$ designates the length of δ. Moreover, $\Phi(\delta)$ is said to satisfy the condition (A) if to each $\sigma>0$ there is a $\lambda_\sigma>0$ such that $\Phi(\delta_1)+\Phi(\delta_2)<\Phi(\delta)+\sigma$ for every interval δ with $|\delta|\leqq\lambda_\sigma$, which is divided into two intervals δ_1 and δ_2. Tonelli [loc. cit.] proved: if $\Phi(\delta)$ is approximately subadditive and satisfies condition (A), then the Burkill integral of Φ on (a, b) (in the proper sense or admitting $+\infty$ as value) exists. According to Tonelli, this is also true if there is a finite number of "exceptional" points. A point p is called "exceptional" if, in every neighborhood of p, $\Phi(\delta)$ satisfies condition (A) without being approximately subadditive, provided that to each $\epsilon>0$ there is a neighborhood U of p such that, for every set of consecutive intervals $\delta_1, \cdots, \delta_n$ of U, $|\Phi(\delta_1)+\cdots+\Phi(\delta_n)|\leqq\epsilon$. The author proves that Tonelli's result still holds if there is a countable set of exceptional points, but that in general Tonelli's result cannot hold for the case of a noncountable set of exceptional points. Moreover, he generalizes the notion of approximately subadditive interval functions, defining "quasi-subadditive" interval functions. For such a quasi-subadditive and absolutely continuous interval function the Burkill integral (with finite value) exists. *A. Rosenthal* (Albuquerque, N. M.).

Fréchet, Maurice. L'intégrale abstraite d'une fonction abstraite d'une variable abstraite et son application à la moyenne d'un élément aléatoire de nature quelconque. Revue Sci. (Rev. Rose Illus.) **82**, 483–512 (1944). [MF 16858]

Various abstract integrals are defined and discussed, culminating in an integral $I=\int\Phi(X)v(de)$ where $v(e)$ is an additive function of sets of points in an abstract space E, with values in a complete normed vector space E', Φ a point function defined on E with values in a complete normed vector space E''; the value of the integral is a point in a complete normed vector space E'''. (Addition and unique subtraction but not multiplication by constants are assumed in E', E'', E'''.) In particular, if $v(e)$ is a nonnegative additive function of sets, with $v(E)=1$, $\Phi(X)$ becomes an abstract chance variable and I becomes its expectation. Following up this idea, generalizations of standard probability theorems are proved with this specialization of $v(e)$. In these applications it is supposed that $E''=E'''$ is a Banach space and, in fact, in developments connected with the law of large numbers, that $E''=E'''$ is a Euclidean space, that is, a Banach space with an inner product. [Cf. also, for the probability theorems under substantially the same hypotheses, Glivenko, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **8**, 673–678 (1928); **9**, 830–833 (1929).] *J. L. Doob* (Urbana, Ill.).

Vidal, Enrique. On rectifiable curves. Revista Mat. Hisp.-Amer. (4) **6**, 132–139 (1946). (Spanish)

Starting with a curve defined by $x=x(t)$, $y=y(t)$ $(t_0\leqq t\leqq t_1)$, where x and y are not assumed continuous, the author gives the usual definition for rectifiability. His result: such a rec-

tifiable curve is either defined by continuous $x(t)$ and $y(t)$ of limited variation or it is a subset of a curve so defined.

J. H. Roberts (Durham, N. C.).

Massera, José L. On Green's formula. Publ. Inst. Mat. Univ. Nac. Litoral 6, 169–178 (1946). (Spanish) [MF 16919]

The elementary proof of Green's lemma in the plane presented in this paper follows the pattern used by the reviewer [Amer. J. Math. 63, 563–574 (1941); these Rev. 3, 75], with the author giving an alternate proof of a central result on the approximation of a simple closed rectifiable plane curve by simple closed polygons interior to this curve. In addition, the usual conditions on the partial derivatives are relaxed somewhat, with the general result stating the equality of a line integral and an iterated integral.

W. T. Reid (Evanston, Ill.).

Adelson-Welsky, G. M., et Kronrode, A. S. Sur les lignes de niveau des fonctions continues possédant des dérivées partielles. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 235–237 (1945).

A level set E_{u_0} (ensemble de niveau) of a real function $u(x, y)$ consists of the points where $u=u_0$. It is assumed that (A) the set G of definition is a simply connected plane region whose boundary C is homeomorphic to a circle; (B) $u(x, y)$ is a real continuous function with minimum m and maximum M, defined in G and possessing first partial derivatives. Among the results are the following. Under the above hypotheses, the segment mN of the u-axis contains a set U of measure $M-m$ such that for each $u\epsilon U$ the level set E_u contains only the following components: (a) simple closed curves; (b) simple arcs with extremities on C; (c) points. For each $u\epsilon U$, the set of points which are components has projections on the x- and y-axes of measure zero. If in addition it is assumed that u has a total differential at each point of G, then for each $u\epsilon U$ the totality of points which are both components and points where E_u has no tangent has projections on the x- and y-axes of measure zero.

A. B. Brown (Flushing, N. Y.).

Martinez Salas, J. Functions of n real variables of bounded variation. Revista Mat. Hisp.-Amer. (4) 6, 25–42 (1946). (Spanish) [MF 16707]

The author defines a function of bounded variation for real-valued functions of n real variables defined in a closed n-dimensional hypercube. These functions preserve some of the important properties of functions of bounded variation of a single real variable. Each such function is the difference of two nondecreasing "monotone" functions. The points of discontinuity are all of the first kind and are contained in a denumerable number of hyperplanes, $x_{im}=$constant, $i=1, \cdots, n$; $m=1, 2, \cdots$. The actual definitions of "bounded variation" and of "monotone" are fairly simple but cannot easily be stated in a brief manner.

J. V. Wehausen (Falls Church, Va.).

Tibaldo, Lina. Sulla differenziabilità quasi regolare delle funzioni di tre variabili. Rend. Sem. Mat. Univ. Padova 13, 78–88 (1942).

If $f(x_1, x_2, x_3)$ is continuous in an open set G of Cartesian 3-space and has first partial derivatives almost everywhere in G, then for almost every point $y=(y_1, y_2, y_3)$ of G there exist constants a_1, a_2, a_3 and a set U which has density 1 at y and is the union of boundaries of cubes parallel to the coordinate planes with center y, such that

$$\lim[f(x_1,x_2,x_3)-f(y_1,y_2,y_3)-\sum a_i(x_i-y_i)]/[\sum(x_i-y_i)^2]^{\frac{1}{2}}=0$$

when $x\rightarrow y$ on U. The corresponding result for two dimensions was proved by Radó [Fund. Math. 30, 34–39 (1938)] and by Cacciopoli and Scorza Dragoni [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (6) 9, 251–268 (1938)].

H. Busemann (Northampton, Mass.).

Cesari, Lamberto. Criteri di uguale continuità ed applicazioni alla quadratura delle superficie. Ann. Scuola Norm. Super. Pisa (2) 12, 61–84 (1943). [MF 16767]

Let $S: x=x(u, v), y=y(u, v), z=z(u, v)$, where x, y, and z are continuous on a closed Jordan region $R+C$, C being the bounding curve. The author defines S to be of type A if no maximal continuum g, over which all three functions are constant, separates the plane; S is said to be nondegenerate if $R-R\cdot g$ is connected for every such continuum g. These properties are shown to be independent of the representation of S. It is also shown that every surface of type A which is bounded by a Jordan curve is nondegenerate. The new results in the paper are the following two equicontinuity theorems. (I) Let $\{S_n\}$ and S all be of type A and suppose $S_n\rightarrow S$ in the sense of Fréchet. Suppose that each S_n is represented on the unit circle by $S_n: x=X_n(u, v), y=Y_n(u, v), z=Z_n(u, v)$, in which X_n, Y_n, and Z_n are absolutely continuous in the sense of Tonelli with uniformly bounded (independent of n) Dirichlet integrals. Then X_n, Y_n, and Z_n are equicontinuous on each interior closed set. (II) If, in (I), we require also that S and S_n all are nondegenerate and also that there are three points P_1, P_2, P_3 on the boundary whose images Q_{1n}, Q_{2n}, Q_{3n} tend to three distinct points Q_1, Q_2, Q_3, then the functions are equicontinuous on the closed unit circle. The author uses these theorems to prove known theorems [proved independently by E. J. McShane and the reviewer] on the area of surfaces, to prove Schwarz's theorem concerning the conformal mapping of nondegenerate polyhedra by variational methods, and, finally, to derive the reviewer's result concerning the (generalized) conformal mapping of an arbitrary nondegenerate surface of finite Lebesgue area [cf. Amer. J. Math. 57, 692–702 (1935)].

C. B. Morrey, Jr. (Berkeley, Calif.).

Cesari, Lamberto. Sulle trasformazioni continue e sull' area delle superficie. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 12, 1305–1397 (1942).

Cesari, Lamberto. Sulle superficie di area finita secondo Lebesgue. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 3, 350–365 (1942).

Cesari, Lamberto. Sui fondamenti geometrici dell'integrale classico per l'area delle superficie in forma parametrica. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 13, 1323–1481 (1943).

Cesari, Lamberto. Una uguaglianza fondamentale per l'area delle superficie. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 14, 891–951 (1944).

The principal notations used by the author are as follows. The symbol Φ refers to a continuous transformation Φ: $x=x(u, v)$, $y=y(u, v)$, $(u, v)\epsilon A$, where A denotes the unit square $0\leqq u\leqq 1$, $0\leqq v\leqq 1$. The letter r serves as a generic notation for a simply connected Jordan region in A. The image of the positively oriented boundary of r, under Φ, is a closed continuous curve C in the (x, y)-plane, and $O(x, y, r)$ denotes the topological index of the point (x, y) relative to C if (x, y) does not lie on C, while $O(x, y, r)=0$ for (x, y) lying on C. The function $o(x, y, r)$ is defined as 1 if

$O(x, y, r) \neq 0$ and as 0 if $O(x, y, r)=0$. The image of A under Φ being a bounded set, we can choose in the (x, y)-plane a square K, with sides parallel to the axes, such that the image of A is interior to K. We put

$$g(r)=\iint_K |O(x, y, r)|\,dxdy, \quad u(r)=\iint_K o(x, y, r)dxdy,$$

$$t(r)=\left|\iint_K O(x, y, r)dxdy\right|,$$

with the understanding that $g(r)=t(r)=+\infty$ if $O(x, y, r)$ is not summable. Now let $r_1, \cdots, r_n$ be any finite system of simply connected Jordan regions in r without common interior points. We define $G(r)=\sup \sum g(r_j)$, $j=1, \cdots, n$, where the least upper bound is taken with respect to all possible systems $r_1, \cdots, r_n$. The quantities $U(r)$, $T(r)$ are defined similarly in terms of $u(r)$, $t(r)$. Now let (u, v) be a point interior to A and let q denote a square in A, with sides parallel to the axes, such that (u, v) is interior to q. Then the limit (if it exists) of $U(q)/|q|$ for $|q|\rightarrow 0$ is denoted by $J(u, v)$ and is termed the generalized Jacobian. The characteristic function $\Psi(x, y, \Phi)$ of the mapping Φ is defined as $\sup \sum |O(x, y, r_j)|$, where the least upper bound is taken with respect to all possible finite systems of simply connected Jordan regions $r_1, \cdots, r_n$, without common interior points, in A. The function $\psi(x, y, \Phi)$ is defined similarly in terms of $o(x, y, r)$. If $\psi(x, y, \Phi)<\Psi(x, y, \Phi)$ at some point (x, y), then (x, y) is termed a branch point relative to Φ. The total variation $W(\Phi)$ is defined as the double integral of $\Psi(x, y, \Phi)$ over K, with the understanding that $W(\Phi)=+\infty$ if $\Psi(x, y, \Phi)$ fails to be summable; Φ is said to be of bounded variation if $W(\Phi)<+\infty$. Now let $\pi_1, \cdots, \pi_n$ be the generic notation for a finite system of simply connected polygonal regions in A without common interior points. Then Φ is termed absolutely continuous if the following two conditions hold. (a) For every $\epsilon>0$ there exists a $\sigma>0$ such that $\sum|\pi_j|<\sigma$ implies that $\sum g(\pi_j)<\epsilon$. (b) If $A=\sum \pi_j$, then $G(\Phi)=\sum G(\pi_j)$. Finally, let $L(\Phi)$ denote the Lebesgue area of the flat surface $x=x(u, v)$, $y=y(u, v)$, $z=0$, $(u, v)\epsilon A$.

In terms of these concepts and definitions, the author discusses a large number of theorems. Some illustrations follow (we have to restrict ourselves to theorems whose statements are relatively simple). The set of branch points of Φ is always countable. If Φ is of bounded variation, $G(\Phi)=U(\Phi)=T(\Phi)=W(\Phi)=L(\Phi)$. If Φ is of bounded variation, then $J(u, v)$ exists almost everywhere in A and $W(\Phi)\geqq \iint_A J(u, v)dudv$, where the sign of equality holds if and only if Φ is absolutely continuous. If Φ is of bounded variation, and if the first partial derivatives x_u, x_v, y_u, y_v exist almost everywhere in A, then $J(u, v)=|x_u y_v - x_v y_u|$ almost everywhere in A. These results are discussed in the third of the four papers.

All four papers contain applications to surface area theory. Let S: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, $(u, v)\epsilon A$, be a representation of a surface S. Let us introduce the three continuous mappings Φ_1: $x=x(u, v)$, $y=y(u, v)$; Φ_2: $x=x(u, v)$, $z=z(u, v)$; Φ_3: $y=y(u, v)$, $z=z(u, v)$, $(u, v)\epsilon A$. Relative to these mappings Φ_i, $i=1, 2, 3$, we then have the quantities $g_i(r)$, $u_i(r)$, $t_i(r)$, $G_i(r)$, $U_i(r)$, $T_i(r)$, $W(\Phi_i)$, $L(\Phi_i)$, $J_i(u, v)$. We define $g(r)$ as the square root of the sum of the squares of $g_1(r)$, $g_2(r)$, $g_3(r)$, and $G(r)$ is then defined as $\sup \sum g(r_j)$, where $r_1, \cdots, r_n$ is any finite system of simply connected Jordan regions in r without common interior points and the least upper bound is taken with respect to all such systems. The quantities $u(r)$, $t(r)$, $U(r)$, $T(r)$ are defined similarly; $L(S)$ denotes the Lebesgue area of S; $D(u, v)$ is the square root of the sum of the squares of $J_1(u, v)$, $J_2(u, v)$, $J_3(u, v)$. In terms of these concepts, the author discusses a number of theorems concerning surface area. A few samples follow. Always $L(S)\leqq L(\Phi_1)+L(\Phi_2)+L(\Phi_3)$, with similar inequalities in which L is replaced by G, U, T, respectively. The area $L(S)<+\infty$ if and only if the mappings Φ_1, Φ_2, Φ_3 are of bounded variation. If $L(S)<+\infty$, then $D(u, v)$ exists almost everywhere in A and $L(S)\geqq \iint_A D(u, v)dudv$. The sign of equality holds if and only if the mappings Φ_1, Φ_2, Φ_3 are absolutely continuous. Always $G(S)=L(S)$.

While the second, third, and fourth papers are concerned with general continuous mappings and surfaces, the first paper is devoted to a special situation. Given Φ, let $C(\xi)$, $C'(\eta)$ denote the images of the segments $u=\xi$, $0\leqq v\leqq 1$ and $0\leqq u\leqq 1$, $v=\eta$, respectively. Then Φ is termed regular if there exist countable everywhere dense sets E, E' in the intervals $0\leqq u\leqq 1$, $0\leqq v\leqq 1$, respectively, such that the two-dimensional measure of $C(\xi)$ is zero for $\xi\epsilon E$ and the two-dimensional measure of $C'(\eta)$ is zero for $\eta\epsilon E'$. The surface S is termed regular if the countable everywhere dense sets E, E' can be so chosen that the condition just stated holds simultaneously for all three mappings Φ_1, Φ_2, Φ_3. The purpose of the discussion of regular surfaces and mappings is to show that the general theory admits of relatively elementary presentation in this special case.

In view of the amount of material included in the papers, the reviewer was unable to study them in adequate detail and to determine the exact relationships between the work of Cesari and recent researches carried on in America.

T. Radó (Columbus, Ohio).

Theory of Functions of Complex Variables

✶Sansone, Giovanni. Lezioni sulla Teoria delle Funzioni di una Variabile Complessa. CEDAM, Padova, 1947. Vol. I, viii+359 pp., 750 lire; vol. II, xi+564 pp., 950 lire.

These two volumes constitute an extensive account of the classical theory of functions of a complex variable. They are based on the author's course given at the University of Florence and presuppose a knowledge of the elements of analysis. Apart from elementary complex function theory considerable space is devoted to topics not always found in texts of this character. One finds a fairly complete elementary account of such subjects as the conformal mapping of simply-connected regions, the Dirichlet problem, two-dimensional hyperbolic geometry, automorphic functions, special classes of functions (hypergeometric, elliptic, elliptic modular, etc.).

The following summary by chapters will indicate the scope of the work. 1. Generalities, power series, elementary functions. 2. Cauchy theory, analytic continuation. 3. Isolated singularities, calculus of residues, theorems of the Rouché type. 4. Weierstrass product theorem, Mittag-Leffler theorem. 5. Entire functions: the theorems of Laguerre, Picard, Poincaré, Hadamard, Borel, Schottky, Landau. 6. Euler-Maclaurin series and related topics. 7. Dirichlet series, the Riemann zeta function, hypergeometric functions. 8. Conformal mapping of plane simply-connected regions, special mapping functions. 9. Harmonic

functions, the Dirichlet and Neumann problems, the logarithmic potential. 10. Two-dimensional hyperbolic geometry, conformal mapping on non-Euclidean polygons. 11. Elliptic functions and integrals. The theories of Weierstrass and Jacobi. 12. Introduction to the theory of automorphic functions and Poincaré series. *M. H. Heins.*

Widder, D. V. A simplified approach to Cauchy's integral theorem. Amer. Math. Monthly **53**, 359–363 (1946).

Cauchy's integral theorem is established for regular closed curves Γ. The author avoids most of the topological difficulties by first showing the existence of the indefinite integral and then using it to compute the integral in the theorem. Thus the usual approximation to the regular curve by a polygonal line is unnecessary. Separate proofs are given under hypotheses that the integrand has a derivative and has a continuous derivative, respectively. Parts of the proofs are standard but are included for completeness.
E. F. Beckenbach (Los Angeles, Calif.).

Rios, Sixto. Prolungamento analitico mediante permutazione dei termini di una serie. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 677–683 (1941).

The author discusses the analytic continuation of a function defined by a Dirichlet series. [Cf. his more recent papers, Revista Mat. Hisp.-Amer. (4) **3**, 100–128 (1943); **4**, 206–209 (1944); Revista Acad. Ci. Madrid **37** (1943); these Rev. **5**, 66; **6**, 210; **7**, 61.] *R. P. Boas, Jr.*

***Netanyahu (Mileikowsky), Elisha. Researches on the singularities of analytic functions represented by multi-Taylor series.** Summary of a thesis, Hebrew University, Jerusalem, 1942. 10+2 pp. (Hebrew. English summary)

Summary of a dissertation concerned with the existence and nature of the singularities of functions admitting the representation

$$F(x)=\sum_{n=0}^{\infty}a_n(x)\{\omega(x)\}^n,$$

where $\omega(x)=x^k+c_1x^{k-1}+\cdots+c_k$ and $a_n(x)=\sum_{j=0}^{k-1}a_n^{(j)}x^j$ $(n\geqq 0)$. *M. H. Heins* (Providence, R. I.).

Walsh, J. L. Note on the location of the zeros of the derivative of a rational function having prescribed symmetry. Proc. Nat. Acad. Sci. U. S. A. **32**, 235–237 (1946).

This paper contains an announcement of some new results connected with Jensen's theorem that all the nonreal zeros of the derivative of a real polynomial $f(z)$ lie in or on the circles (Jensen circles) whose diameters are the line-segments joining the pairs of conjugate imaginary zeros of $f(z)$. First, for the case that $f(z)$ has only one pair of conjugate imaginary zeros α and $\bar{\alpha}$ and has all its real zeros on the interval $x_0\leqq z\leqq x_1$, all the nonreal zeros of $f'(z)$ lie in the closed lens-shaped region bounded by the two circular arcs α, $\bar{\alpha}$ tangent to the lines (x_0, α) and $(x_1, \bar{\alpha})$. Second, for a real rational function $F(z)$ which has no poles interior to the unit circle C and whose zeros are inverse to the poles with respect to C, the nonreal zeros of $F'(z)$ interior to C lie in or on the non-Euclidean Jensen circles whose non-Euclidean diameters are the non-Euclidean segments joining pairs of conjugate imaginary zeros of $F(z)$. Third, for a rational function $F(z)$ whose poles are symmetric to its zeros in the origin O, all the zeros of $F'(z)$ lie in the double sector whose vertex is at O and one of whose halves contains all the zeros of $F(z)$ and the other all the poles of $F(z)$. Proofs are to be published later. *M. Marden.*

Walsh, J. L. Taylor's series and approximation to analytic functions. Bull. Amer. Math. Soc. **52**, 572–579 (1946).

Der Verfasser formuliert und beweist die folgenden Theoreme. Die Taylor-Reihe $f(z)=a_0-a_1z+a_2z^2+\cdots$ besitze den Konvergenzkreis $|z|<R$ $(R>1)$. Es sei $R_0>R$ und M eine beliebige positive Zahl; $f_M(z)$ sei regulär für $|z|<R_0$, wobei $|f_M(z)|<M$ und $m_M=\max|f(z)-f_M(z)|$ für $|z|=1$ möglichst klein ist. Es gilt

$$\limsup_{M\to\infty} m_M^{1/\log M}=\exp\{(-\log R)/(\log R_0-\log R)\}.$$

Wenn $g_M(z)$ für $|z|<R_0$ regulär ist und $|g_M(z)|<M$, wobei $\mu_M=\max|f(z)-g_M(z)|$ für $|z|=1$, dann gilt die Ungleichung

$$\limsup_{M\to\infty} \mu_M^{1/\log M}\geqq\exp\{(-\log R)/(\log R_0-\log R)\}.$$

Der Verfasser schildert die möglichen Verallgemeinerungen dieser beiden Theoreme und die ungelösten Fragen, die sich daran anschliessen. *W. Saxer* (Zürich).

Ibraguimov, I. Sur les critères pour que la suite des dérivées d'une fonction analytique forme un système complet. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 389–390 (1946).

The author announces three theorems. The first is that, if $\varphi(x+iy)$ is analytic in $|y|<\pi$ and of period 2π, then $\{\varphi^{(n)}(z)\}$ is complete in $|z|<\pi-\epsilon$ if no Fourier coefficient of $\varphi(z)$ is zero. The second is that under the same conditions the set $\{\varphi(z+a_n)\}$ is complete if $|a_n|<1$ and $\lim|a_n|=0$. The third is a condition for the completeness of $\{\varphi(z+a_n)\}$ when $\varphi(z)$ is entire of finite order and periodic.
R. P. Boas, Jr. (Providence, R. I.).

Boas, R. P., Jr. The rate of growth of analytic functions. Proc. Nat. Acad. Sci. U. S. A. **32**, 186–188 (1946). [MF 16860]

"This note presents a theorem of the Phragmén-Lindelöf type and indicates how it can be applied to establish and generalize results of N. Levinson on the determination of the rate of growth of an analytic function along a line from its growth on a sequence of points." The theorem is as follows. Let $\delta(r)$ be a continuous function such that $0\leqq\delta(r)<\frac{1}{2}$ and $\int^{\infty}r^{-1}\delta(r)dr$ diverges. Let $H(z)$ be analytic in $x\geqq 0$ and let it satisfy $\log|H(re^{i\theta})|=o\{\omega(r)\}$, $r\to\infty$, $\omega(r)=r\exp\{\int_{r_0}^{r}s^{-1}\delta(s)ds\}$. Let $H(z)$ be bounded on the curve $x=r\delta(r)$. Then $H(z)$ is bounded in $x\geqq r\delta(r)$. [In the paper the factor r was omitted from the definition of $\omega(r)$.]
N. Levinson (Cambridge, Mass.).

Bernstein, S. Sur la borne supérieure du module de la dérivée d'une fonction de degré fini. C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 567–568 (1946).

In the author's book [Leçons sur les Propriétés Extrémales, Paris, 1926] the following problem is solved: to find, among the entire functions of degree p $(0<p<\infty)$, real valued on the real axis, the one for which

$$\sup_{-\infty<x<\infty}|\varphi_p(x)|=\text{minimum}$$

while $\varphi_p'(0)=1$. Here the same problem is solved for the subclass of entire functions of degree p monotonic in

★ The inequality in line 11 of the Review contradicts a result of J. E. Littlewood [Quart. Jour. Math., Oxford Ser. 9, 14–20 (1938).] There is an error in Biernacki's proof: on page 209, line 16, the exponent $-1/2n$ should read $-1/(nk)$. This vitiates Biernacki's arguments.

$(-\infty, \infty)$. The minimum is attained by

$$\varphi_p(z)=\int_0^z (2(pt)^{-1}\sin \tfrac{1}{2}pt)^2 dt.$$

The proof is based on the representation of an entire function nonnegative on the real axis by the square of the modulus of another entire function, and on Fourier transforms. It is furthermore shown that the conditions (i) $f(z)$ is entire, of degree p, (ii) $f(x)$ is real and monotonic in $(-\infty, \infty)$, (iii) $|f(x)|\leqq M$ in $(-\infty, \infty)$ imply that $|f'(x)|\leqq Mp/\pi$. In the book it was proved that (i) and (iii) alone imply that $|f'(x)|\leqq Mp$. [In the book the degree was defined as $p=\limsup na_n^{1/n}$ as $n\to\infty$ $(f(z)=\sum a_n z^n)$; here the meaning is apparently $p=e^{-1}\limsup na_n^{1/n}$, i.e., p is the type of the entire function of exponential type.]

H. Kober (Birmingham).

Ferrand, Jacqueline, et Dufresnoy, Jacques. Extension d'une inégalité de M. Ahlfors et application au problème de la dérivée angulaire. Bull. Sci. Math. (2) 69, 165–174 (1945).

The reviewer's second inequality on the conformal mapping of strip regions is improved and applied to the existence of an angular derivative. *L. Ahlfors.*

Cisotti, Umberto. Corrispondenza conforme tra campi complementari. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 278–286 (1943).

Let L be a simple curve dividing the z-plane into two domains C and C_1. Let $Z=g(z)$ and $Z=g_1(z_1)$ be conformal mappings of C and C_1 on $|Z|<1$ and $|Z|>1$, respectively. Then $g(z)g_1(z_1)=1$ defines a conformal mapping of C onto C_1. Explicit formulae are given in the case that L is an ellipse. *W. H. J. Fuchs* (Swansea).

Cisotti, Umberto. Funzioni analitiche complementari. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 388–397 (1943).

Using the "complementary functions" $g(z)$ and $g_1(z_1)$ of the preceding review the author finds formulae expressing a function regular in a domain in terms of the boundary values of its real part. [See Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, v. 1, Springer, Berlin, 1925, chap. III, problems 177, 231.] *W. H. J. Fuchs.*

Dedecker, Paul. Pseudo-surfaces de Riemann et pseudo-involutions. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 30 (1944), 120–133 (1945).

This note sketches a theory of pseudo-Riemann surfaces, that is, one in which the magnitude of angles is an invariant concept, while no demand is made on sense. Realizations are given. *M. H. Heins* (Providence, R. I.).

Dedecker, Paul. Pseudo-surfaces de Riemann et pseudo-involutions. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 30 (1944), 179–188 (1945).

Classical results of the theory of Riemann surfaces are extended to the author's "pseudo-surfaces" [see the preceding review]. *M. H. Heins* (Providence, R. I.).

***Weissbach, Willi (Nehari, Zeev). On certain classes of analytic functions and the corresponding conformal representations.** Summary of a thesis, Hebrew University, Jerusalem, 1941. 12+i pp. (Hebrew. English summary)

This paper summarizes the author's dissertation, which is concerned with functions $f(z)=z+a_2z^2+\cdots$, analytic for $|z|<1$ and satisfying further conditions (restrictions on the Riemannian image of $|z|<1$ with respect to f).

M. H. Heins (Providence, R. I.).

Biernacki, M. Sur les fonctions univalentes et K-symétriques. Bull. Sci. Math. (2) 69, 204–214 (1945).

The author studies the class of k-symmetric functions, that is, functions with the expansion

$$f(z)=z+a_1z^{k+1}+\cdots+a_nz^{nk+1}+\cdots,$$

which are analytic and univalent in the circle $|z|<1$. A well-known conjecture of Szegö states that $|a_n|\leqq A(k)n^{2/k-1}$, where $A(k)$ depends on k alone. This has been proved by Littlewood for $k=1$ [Proc. London Math. Soc. (2) 23, 481–519 (1925), in particular, p. 498], by Littlewood and Paley for $k=2$ [J. London Math. Soc. 7, 167–169 (1932)], and by V. Levin for $k=3$ [Math. Z. 38, 306–311 (1934)]. The author shows that $|a_n|\leqq A(k,\epsilon)n^{2/k-1+\epsilon}$ for any $\epsilon>0$, where $A(k,\epsilon)$ depends only on k and ϵ. He also considers mean p-valent functions $f(z)$ analytic in $|z|<1$, that is, functions $f(z)$ such that, if $n(R,\varphi)$ denotes the number of zeros (assumed finite) of $f(z)-Re^{i\varphi}$ in $|z|<1$, $(2\pi)^{-1}\int_0^{2\pi}n(R,\varphi)d\varphi\leqq p$ for all $R\geqq 0$ and p integral. This definition of mean p-valence is less general than that given by D. C. Spencer [Proc. London Math. Soc. (2) 47, 201–211 (1941); these Rev. 3, 79]. The author shows that if $f(z)=z^p+a_1z^{k+p}+\cdots+a_nz^{nk+p}+\cdots$, where p and k are relatively prime and $k<2p$, is analytic and mean p-valent in $|z|<1$, then $|a_n|<A(k,p)n^{2p/k-1}$, $n\geqq 2$. *W. Seidel* (Rochester, N. Y.).

Wolff, J. Domaines d'univalence et d'étoilement des fonctions holomorphes à partie réelle positive dans un demi-plan. Nederl. Akad. Wetensch., Proc. 44, 1210–1213 (1941). [MF 15770]

If $w(z)$ is analytic and has a nonnegative real part in the right half-plane $x>0$, then $w(z)$ can be written as $\lambda z+H(z)$, where $H(z)$ is analytic with nonnegative real part and $H(z)/z\to 0$ as $z\to\infty$. The product $zH(z)$ tends to a positive or infinite limit μ as $z\to\infty$. The author considers the class $C(\lambda,\mu)$ of such functions $w(z)$ for which λ and μ are fixed, positive and finite. The function $w(z)$ is called star-shaped with respect to $x=\infty$ in the half-plane $x>x_0$ if the image of every vertical line $x=a>x_0$ is a curve in the w-plane which cuts each horizontal line at most once. Star-shapedness implies univalence. Using the Cauchy-Stieltjes integral representation for $H(z)$, the author proves that every function of $C(\lambda,\mu)$ is star-shaped in the half-plane $x>(\mu/\lambda)^{\frac{1}{2}}$ and that the only functions of $C(\lambda,\mu)$ for which this half-plane is the greatest half-plane of univalence, hence of star-shapedness, are $w(z)=\lambda z+\mu/(z-ki)$, $-\infty<k<\infty$.

L. H. Loomis (Cambridge, Mass.).

Fried, Hans. On analytic functions with bounded characteristic. Bull. Amer. Math. Soc. 52, 694–699 (1946).

A function $f(z)=f(re^{i\varphi})$, regular in $|z|<1$, has bounded characteristic A if

$$A=\lim_{r\to 1}\int_0^{2\pi}\log^+|f(re^{i\varphi})|d\varphi<\infty.$$

The boundary values $f(e^{i\varphi})=\lim_{r\to 1}f(re^{i\varphi})$ then exist for almost all φ. The main theorem is as follows. Suppose that $f(z)$ and the functions $f_n(z)$ are of bounded characteristic. If (i) $|f(e^{i\varphi})-f_n(e^{i\varphi})|<m_n$ for all φ of a set E_n of measure μ_n, such that $m_n^{\mu_n}\to 0$; (ii) $A_n<m_n^{-\sigma\mu_n}$ for every positive σ and

all $n>n_\epsilon$, then $f_n(z)\to f(z)$ uniformly in any closed domain in $|z|<1$. This theorem contains older results by Ostrowski [Acta Litt. Sci. Szeged 1, 80–87 (1923)] and Milloux [J. Math. Pures Appl. (9) 3, 345–401 (1924)]. It is also proved that, for a gap power series $\sum c_k z^{n_k}$, $n_{k+1}/n_k \geqq q>1$, of bounded characteristic, the series $\sum|c_k|^2$ converges, and that the derivative even of a bounded function need not be of bounded characteristic. *W. W. Rogosinski.*

Bergman, Stefan. Models in the theory of several complex variables. Amer. Math. Monthly 53, 495–501 (1946).

Terracini, Alejandro. A first contribution to the geometry of monodiffric polynomials. Actas Acad. Ci. Lima 8, 217–250 (2 plates, 1 p. errata) (1945). (Spanish)

In a recent article [Univ. Nac. Tucumán. Revista A. 2, 177–201 (1941); these Rev. 3, 298], R. P. Isaacs considered a class of functions of a complex variable which are analogous to analytic functions. This is the class of monodiffric functions defined by the equality of the difference quotients $\Delta f(z)=f(z+1)-f(z)=(1/i)\{f(z+i)-f(z)\}$. In particular, the monodiffric polynomials of degree n were considered. These can be expressed in terms of the pseudopowers $z^{(n)}$ defined by the recursive relation $\Delta z^{(n)}=nz^{(n-1)}$ and $z^{(0)}=1$. The author now submits a development of the theory of monodiffric equations. He proves that a monodiffric polynomial of degree n has n^2-2m complex roots, where $0\leqq m\leqq n(n-1)/2$. Thus the number of roots can be at least n and at most n^2.

In particular, the equations of second degree are discussed, where the number of roots is two, three or four. In the case of four roots, these are found to be the vertices of a rectangle and are the intersections of two equilateral hyperbolas. This rectangle is the limiting case of a pencil of equilateral hyperbolas whose centers describe the circle of Feuerbach and whose asymptotes are tangent to the hypocycloid of Steiner. The radius of the Feuerbach circle is $\frac{1}{4}\sqrt{2}$ and the three oriented cuspidal tangents of the hypocycloid of Steiner have inclinations of 135°, −15° and 105°. This is always the case for the pencil of equilateral hyperbolas to be associated with the four roots of a monodiffric quadratic equation.

The other cases where the monodiffric quadratic equation has two and three complex roots are discussed in detail. In the final part of the paper an application of these results is made to monodiffric cubic equations. *J. DeCicco.*

Volkof, D. Les fonctions analytiques dans le champ des nombres hypercomplexes. Leningrad State Univ. Annals [Uchenye Zapiski] 83 [Math. Ser. 12], 92–113 (1941). (Russian. French summary) [MF 16491]

The author considers analytic functions of expressions of the form $u=\sum_{r=0}^{n-1}u_r(x,y)j^r$, where j satisfies the equation $\sum_{r=0}^{n}A_rj^r=0$, A_r being constants. Operations with such functions, in particular, integration and differentiation, are defined. A generalized Cauchy formula is introduced. The components of the functions satisfy systems of linear partial differential equations. In particular, it is possible to obtain solutions of the Laplace and biharmonic equations; the author obtains general formulas for the representation of their solutions. These formulas can be used for the solution of boundary value problems. The reviewer remarks that since the j's are algebraic numbers the question of uniqueness of the representation of the functions should be clarified. *S. Bergman* (Cambridge, Mass.).

Theory of Series

Wendelin, H. Verallgemeinerung der bekannten Beziehungen zwischen den Grenzen und Limites der Folgen (x_n), (y_n) und (x_n+y_n), $(x_n\cdot y_n)$. Deutsche Math. 7, 204–205 (1943).

Wendelin, H. Einheitliche Ableitung der bekannten Beziehungen zwischen den Grenzen und Limites der Folgen (x_n), (y_n), (x_n+y_n) und $(x_n\cdot y_n)$. Deutsche Math. 6, 265–266 (1941).

Babini, José. On a class of developments of the number e in series. Math. Notae 6, 40–44 (1946). (Spanish)

The paper describes a procedure for obtaining expansions of the form

$$e=k^{-1}\sum_0^\infty\frac{1}{n!f(n)f(n+1)},$$

where k is rational and $f(x)$ is a polynomial, of arbitrary degree p, which is found by solving a second order linear difference equation. Explicit determinantal formulas are given for $f(n)$ and $f(n+1)$. The method for finding k is contingent on knowing the roots of $f(x)$, which are assumed to be distinct (although there is no proof that this is always the case). The results given are not valid if $p=1$. *A. E. Taylor* (Los Angeles, Calif.).

Oguiewetzki, I. E. Un théorème sur les séries fonctionnelles présentant une généralisation d'un théorème de Landau. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 667–669 (1946).

In order that $\sum b_n(x)S_n(x)$ converge uniformly over $a\leqq x\leqq b$ whenever $S_n(x)$ is a sequence of functions uniformly bounded over the interval, it is necessary and sufficient that $\sum a_n(x)u_n(x)$ converge uniformly over the interval when $a_n(x)=b_{n+1}(x)+b_{n+2}(x)+\cdots$ and whenever $u_n(x)=T_{n+1}(x)-T_n(x)$, where $T_n(x)$ is a sequence of functions uniformly bounded over the interval. *R. P. Agnew* (Ithaca, N. Y.).

Macphail, M. S. Euler-Knopp summability of classes of convergent series. Amer. J. Math. 68, 449–450 (1946).

A given series $\sum u_k$ with partial sums s_n is said to be summable to S by means of the Euler-Knopp method $E(r)$ if $S_n\to S$ as $n\to\infty$, where

$$S_n=\sum_{k=0}^n\binom{n}{k}r^k(1-r)^{n-k}s_k,$$

r being real or complex. If $\sum u_k$ converges denote its sum by s. It is known that $S_n\to s$ for every convergent series, if and only if $0<r<1$. It is shown that, if $R>1$, a necessary and sufficient condition for $E(r)$ to have the property that $\sum u_k$ is summable $E(r)$ to s whenever $\sum u_kz^k$ has its radius of convergence not less than R is that $|r/R|+|1-r|<1$. *J. D. Hill* (East Lansing, Mich.).

Erdös, P., and Rosenbloom, P. C. Toeplitz methods which sum a given sequence. Bull. Amer. Math. Soc. 52, 463–464 (1946). [MF 16802]

The authors prove the following theorem and corollary. Let $\{x_n\}$ be a bounded divergent sequence and suppose that $\{y_n\}$ is summable by every regular Toeplitz method which sums $\{x_n\}$. Then $\{y_n\}$ is of the form $\{cx_n+a_n\}$, where $\{a_n\}$ is convergent. As a consequence, if $\{x_n\}$ and $\{y_n\}$ are bounded divergent sequences such that $\{y_n\}$ is summable by every regular Toeplitz method which sums $\{x_n\}$, then $\{x_n\}$ is summable by every regular Toeplitz method which sums $\{y_n\}$. *J. D. Hill* (East Lansing, Mich.).

Cesco, R. P. On the general theory of the linear methods of summation. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. No. 188, Vol. 3, num. 5. Serie segunda, 15, Revista, 500–516 (1946). (Spanish)

This is an exposition giving references and proofs for many known theorems of the following type. Stated conditions are necessary and sufficient that a square matrix a_{nk} be such that the transform $t_n=\sum_{k=0}^{\infty}a_{nk}s_k$ has a stated property whenever the sequence s_k belongs to a stated class.
R. P. Agnew (Ithaca, N. Y.).

González, Mario O. Essay on divergent series. Universidad de la Habana 9, nos. 52–53–54, 259–276; nos. 55–56–57, 193–220 (1944); nos. 58–59–60, 245–254; 10, nos. 61–62–63, 359–375 (1945).

The first chapter is a historical introduction; the second gives an account of various classical summation methods. The third chapter discusses the method of summation in which $\sum_0^{\infty}a_n$ is summed by considering the power series $f(z)=\sum_0^{\infty}u_nz^n$, $a_n=u_nz_0^n$, and associating with it as sum the generalized value $f(z_0)$ as defined by the process of continuation introduced by the author in Publ. Inst. Mat. Univ. Nac. Litoral 5, 231–244 (1945) [cf. these Rev. 6, 210].
R. P. Boas, Jr. (Providence, R. I.).

Cesari, Lamberto. Sul campo totale di convergenza delle serie doppie di potenze. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 14, 603–616 (1943).

If a double series $\sum A_{mn}$ converges, converges by rows, and converges by columns, then the Abel (or Poisson or Euler) transform $\sum A_{mn}z^mw^n$, $|z|<1$, $|w|<1$ converges uniformly (in both double-limit and iterated-limit senses) over closed regions obtained by restricting z and w to Stolz sectors in the z and w complex planes. Some theorems on Cesàro and Abel summability of double series having bounded partial sums are given with proofs and references to papers dated 1942 and later. These theorems are corollaries of theorems of Robison [Trans. Amer. Math. Soc. 28, 50–73 (1926)] and were used much earlier than 1942 by C. N. Moore and other writers. *R. P. Agnew.*

Rényi, Alfred. On a Tauberian theorem of O. Szász. Acta Univ. Szeged. Sect. Sci. Math. 11, 119–123 (1946).

The author observes that the Szász condition that $n^{-1}\sum_1^n k^p|a_k|^p$ is bounded for all n is a sufficient Tauberian condition, together with the Abel summability of the series $\sum a_k$, to ensure its convergence provided that $p>1$, but not if $p=1$. In the latter case, he shows that a sufficient Tauberian condition is that $\sigma_n^r-\sigma_n^{r+1}$ tends to a limit as $n\to\infty$, where σ_n^r is the Cesàro mean of order r of $\sum|a_k|$. In particular, a sufficient condition is that $n^{-1}\sum_1^n k|a_k|$ tends to a limit as $n\to\infty$. *H. R. Pitt* (Belfast).

Shtshegloff, M. On some problems of summation by Poisson's method. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 423–428 (1945). (Russian. English summary)

[In the original, the author's name was transliterated Chthegloff. The Russian spelling is Ščeglov.] If $\sum a_nx^n$ converges for $|x|<1$, the limits of indetermination of the function $\sum a_nx^n$, as $x\to 1$, are contained between the limits of indetermination of the sequence of the partial sums of the series $\sum a_n$. If $a_n=o(1/n)$, the two segments of indetermination coincide [Littlewood, Proc. London Math. Soc. (2) 9, 434–448 (1911)]. The author gives examples illustrating situations when $a_n=O(1/n)$. *A. Zygmund.*

Shtsheglov, M. On convergence and boundedness of Dirichlet's series. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 527–530 (1945). (Russian. English summary)

Typical result. Suppose that a Dirichlet series $\sum a_ne^{-\lambda_nt}$, with $\lambda_{n+1}=O(\lambda_n)$ converges for $t>0$. Let $t_1>\cdots>t_m\to 0$, $t_m-t_{m+1}=O(t_{m+1})$, $a_n<o(\lambda_{n+1}-\lambda_n)\lambda_n^{-1}$. Then $f(t_m)\to 0$ implies $f(t)\to 0$ as $t\to+0$. *A. Zygmund* (Philadelphia, Pa.).

Agnew, Ralph Palmer. Tauberian theorems for Nörlund summability. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. No. 188, Vol. 3, num. 5. Serie segunda, 15, Revista, 517–520 (1946).

A series $\sum u_n$ with partial sums s_k is summable to sum s by the Nörlund method (N, p_n) if $p_n>0$, $P_n=p_0+p_1+\cdots+p_n$ and

$$\lim_{n\to\infty} P_n^{-1}\sum_{k=0}^{n}p_{n-k}s_k=s.$$

The author proves that if np_n/P_n is bounded, Nörlund summability, together with the Tauberian condition $P_nu_n/p_n>-K$, implies convergence. The theorem generalizes one due to R. P. Cesco [same Publ. No. 180, vol. 3, num. 4. Serie Segunda, 14, Contribuciones, 443–445 (1944); these Rev. 6, 210] and is deduced from Littlewood's converse of Abel's theorem [Proc. London Math. Soc. (2) 9, 434–448 (1911)] and the theorem of Silverman and Tamarkin [Math. Z. 29, 161–170 (1928)] that Abel's method of summation is stronger than Nörlund's. *H. R. Pitt.*

Minakshisundaram, S., and Rajagopal, C. T. On a Tauberian theorem of K. Ananda Rau. Quart. J. Math., Oxford Ser. 17, 153–161 (1946).

The main results are Tauberian theorems for Rieszian means, on the following pattern. Let $0<\lambda_0<\lambda_1<\cdots$, $\lambda_n\to\infty$,

$$A(\omega)=\sum_{\lambda_n\leq\omega}a_n,\quad A^r(\omega)=\sum_{\lambda_n\leq\omega}(\omega-\lambda_n)^ra_n,\quad r>0.$$

If (i) $a_n/(\lambda_n-\lambda_{n-1})=O[\psi(\lambda_n)]$, and (ii) $A^r(\omega)=O[\varphi(\omega)]$, where ψ and φ are suitably restricted positive functions, then $A(\omega)=O[\vartheta(\omega)]$, where $\vartheta^{r+1}=\psi^r\varphi$. Here O may be replaced by o in (i) or (ii) and in the conclusion. There is also a "high indices theorem" in which (i) is replaced by $\liminf \lambda_{n+1}/\lambda_n>1$, and the conclusion by $A(\lambda_n)=O[\varphi(\lambda_n)/\lambda_n^r]$ (or $o[\varphi(\lambda_n)/\lambda_n^r]$). There are also theorems with one-sided O or o conditions, and theorems with an averaged O condition on a_n in place of (i). (The conditions on ψ and φ may vary in detail from one theorem to another.) References are given to related theorems due to a number of other authors. The method of proof is modelled on one used by Bosanquet.
A. E. Ingham (Cambridge, England).

Ingham, A. E. Some Tauberian theorems connected with the prime number theorem. J. London Math. Soc. 20, 171–180 (1945).

The author uses Wiener's general Tauberian theorem to prove the following special Tauberian theorems. (I) Suppose that (i) $f(x)$ is positive and increasing for $x\geqq 1$;

(ii) $$F(x)=\sum_{n\leq x}f(x/n)=ax\log x+bx+o(x)$$

when $x\to\infty$, where a and b are constants. Then (a) $f(x)\sim ax$ when $x\to\infty$; (b) $\int_1^{\infty}x^{-2}\{f(x)-ax\}dx$ converges to $b-a\gamma$, where γ is Euler's constant. (II) Suppose that (i) $a_n>-K/n$, where K is a positive constant and $n=1, 2, \cdots$; (ii) $\sum_{n\leq x}a_n(n/x)[x/n]\to A$ when $x\to\infty$. Then $\sum_{n=1}^{\infty}a_n$ converges to sum A.

Theorem (II) is deduced from (I) by putting

$$f(x)=\sum_{n\leqq x} na_n+K[x].$$

The prime number theorem, together with the group of theorems equivalent to it, follows in a particularly simple and direct way from (I(a)) with $f(x)=\psi(x)$ in the usual notation. When the Tauberian condition (i) is dropped in theorem (I), the conclusion (b) remains true and the analogous conclusion that $\sum_{n=1}^{\infty}a_n+A$ $(C, 1)$ is also true when the Tauberian condition (i) is dropped in theorem (II).

The author remarks, finally, that the method of summation defined by theorem (II(ii)) is intermediate in strength between $(C, -\delta)$ and (C, δ) for any positive δ, but is not directly comparable with ordinary convergence.

H. R. Pitt (Belfast).

Persson, Karl. Sur un système d'équations linéaires. Ark. Mat. Astr. Fys. **32A**, no. 12, 8 pp. (1945).

A theorem of W. M. Shepherd and the reviewer [Quart. J. Math., Oxford Ser. **10**, 1–10 (1939)] is extended by an application of complex function theory. The extended theorem can be stated as follows. If λ is complex and not an integer, the system

$$\sum_{q=1}^{\infty} x_q/(p-\theta-\lambda)=0, \qquad p=1, 2, \cdots,$$

of linear equations in the variables $x_1, x_2, \cdots$ has only the trivial solution $x_1=x_2=\cdots=0$ in the case $\Re(\lambda)\geqq 0$. In the case $\Re(\lambda)<0$, the general solution is

$$x_q=\binom{\lambda+q-1}{q-1}\{c_0+c_1(\lambda+q)+\cdots+c_s(\lambda+q)^s\}, \qquad q=1, 2, \cdots,$$

where $c_0, c_1, \cdots, c_s$ are arbitrary constants and $-s$ is the greatest integer not exceeding $\Re(\lambda)+1$. *E. H. Linfoot.*

Davis, C. S. On some simple continued fractions connected with e. J. London Math. Soc. **20**, 194–198 (1945).

The author obtains the regular continued fraction expansions for exp $(1/k)$ and exp $(2/k)$, due to Euler and Stieltjes [Perron, Die Lehre von den Kettenbrüchen, Teubner, Leipzig-Berlin, 1913, pp. 134, 138] by an unusual method suggested by Hermite's proof of the transcendence of e. Rational approximations are first obtained in terms of definite integrals. These turn out to be the convergents by reason of the goodness of fit. The partial quotients are then obtained from recurrence relations between the definite integrals. *D. H. Lehmer* (Berkeley, Calif.).

Wall, H. S. A theorem on arbitrary J-fractions. Bull. Amer. Math. Soc. **52**, 671–679 (1946).

Let $(X_p(z))/(Y_p(z))$ be the $(p-1)$th approximant of the J-fraction $\mathrm{K}_{n=0}^{\infty}(-a_n^2/(b_{n+1}+z))$, $-a_0^2=1$. It is shown that if $(X_n(0))^2$ and $(Y_n(0))^2$ both converge and if the J-fraction converges for one value of z, then it converges to a meromorphic function of z in the whole z-plane except at the poles. Convergence of S-fractions $\mathrm{K}_{n=1}^{\infty}(1/k_nz^{a_n})$, where $a_n=1$ for n odd, $a_n=0$ for n even, is also studied.

W. J. Thron (St. Louis, Mo.).

Wall, H. S. Reciprocals of J-matrices. Bull. Amer. Math. Soc. **52**, 680–685 (1946).

A J-matrix is a matrix associated with a J-fraction [see the preceding review]; J-matrices and J-fractions were studied by Hellinger and Wall [Ann. of Math. (2) **44**, 103–127 (1943); these Rev. **4**, 244]. In this paper theorems on the existence of reciprocals of J-matrices proved by Hellinger [Math. Ann. **86**, 18–29 (1922)] for real a_p and b_p are generalized to the case where the J-matrices are positive definite [Wall and Wetzel, Trans. Amer. Math. Soc. **55**, 373–392 (1944); these Rev. **6**, 151]. *W. J. Thron.*

Wall, H. S. Bounded J-fractions. Bull. Amer. Math. Soc. **52**, 686–693 (1946).

A J-fraction (1) $\mathrm{K}_{n=0}^{\infty}(a_n^2/(b_{n+1}+z))$ is called bounded if there exists an $M>0$ such that $|a_n|\leqq M/3$, $|b_n|\leqq M/3$ for all n. From the Worpitzky criterion it follows easily that (1) converges for $|z|\geqq M$. ~~In this paper it is shown that there exists a convex set in $|z|\leqq M$ such that (1) converges for z outside the set.~~ *W. J. Thron* (St. Louis, Mo.).

Replaced this sentence by what is found in Errata p. 708.

Fourier Series and Generalizations, Integral Transforms

van der Corput, J. G., and Visser, C. Inequalities concerning polynomials and trigonometric polynomials. Nederl. Akad. Wetensch., Proc. **49**, 383–392 = Indagationes Math. **8**, 238–247 (1946). [MF 16821]

This note continues an earlier one by Visser [same Proc. **47**, 276–281 = Indagationes Math. **7**, 81–86 (1945); these Rev. **7**, 440]. If $F(t)=\sum a_k e^{ikt}$ is a finite trigonometric sum then the authors show that

$$\max|F(t)|\geqq\max\Big|\sum_{k\equiv m\,(\mathrm{mod}\ s)} a_k e^{ikt}\Big|,$$

$$\int_0^{2\pi}|F(t)|dt\geqq 2s\max\Big|\sum_{k\equiv m\,(\mathrm{mod}\ s)}\frac{a_k}{-\frac{1}{2}s-m+k}e^{ikt}\Big|.$$

Using these inequalities as starting point they prove a number of inequalities for the coefficients in a trigonometric sum. They show among other things that, if $F(t)=\sum_{-n}^{n}a_k e^{ikt}$, then for $k>\frac{1}{2}n$,

$$|a_{-k}|+|a_k|\leqq\max|F(t)|,$$

$$|a_{-k}|+|a_k|\leqq\tfrac{1}{4}\int_0^{2\pi}|F(t)|dt$$

and, if $k>\frac{1}{3}n$, $|a_0|+|a_{-k}|+|a_k|\leqq\max|F(t)|$. These include as special cases the inequalities given in the earlier paper by Visser and the first includes the inequalities of Chebyshev [see the review cited above] and Markoff [Math. Ann. **77**, 213–258 (1916)]. *A. C. Offord.*

Zahorski, Zygmunt. Un problème de la théorie des ensembles et des fonctions. C. R. Acad. Sci. Paris **223**, 465–467 (1946).

(1) As is well known, the problem of the convergence of Fourier series of functions of the class L^2 is still unsolved. The author constructs a number D, $0\leqq D\leqq 1$, with the following property [no proofs are given]. If $D=0$, the Fourier series of every function of the class L^2 converges almost everywhere. If $D=1$, there is a quadratically integrable function with Fourier series diverging almost everywhere. The definition of D uses only determined constants and is almost one page long. Nothing is known about the numerical value of D. (2) The author gives properties of sets where an arbitrary function is nondifferentiable.

A. Zygmund (Philadelphia, Pa.).

Lozinski, S. On a theorem of N. Wiener. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 542–545 (1945).

Let $f(x)$ be a function of period 2π, of bounded variation and such that at every point of discontinuity $f(x)$ is con-

tained between its limits from the right and from the left. Let $\rho_k=(a_k^2+b_k^2)^{\frac{1}{2}}$, where a_k, b_k are the Fourier coefficients of f. The well-known result of Wiener [J. Math. Phys. Mass. Inst. Tech. 3, 72–94 (1924)] asserts that the condition (*) $\sum_1^n k\rho_k=o(n)$ is both necessary and sufficient for the continuity of f. The author now shows that the condition (**) $\sum_1^n \rho_k=o(\log n)$ is necessary and sufficient for the continuity of f. [Obviously, (**) follows from (*) by summation by parts. That (*) is also sufficient is contained in the following well-known result of Lukács [J. Reine Angew. Math. 150, 107–112 (1920)] to the effect that for any (merely integrable) f,

$$(\log n)^{-1}\sum_1^n(a_k \sin kx-b_k\cos kx)\rightarrow-\pi^{-1}\{f(x+0)-f(x-0)\}$$

at every point x at which the right side exists.]

A. Zygmund (Philadelphia, Pa.).

Lozinski, S. On convergence in mean of Fourier series. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 7–10 (1946).

Let Ω be the class of functions $M(u)$, $0\leqq u<\infty$, satisfying the following conditions: (1) $M(0)=0$, $M(u)$ is convex and increasing; (2) $M(u)/u\rightarrow 0$ as $u\rightarrow 0$ and $M(u)/u\rightarrow\infty$ as $u\rightarrow\infty$; (3) $M(2u)=O(M(u))$ as $u\rightarrow\infty$. If $M\epsilon\Omega$, let L^M be the class of functions $f(x)$, defined in $(0, 2\pi)$, measurable and such that $M(|f(x)|)$ is integrable. Let $s_n[f]$ be the partial sums of the Fourier series of a function f. If $M\epsilon\Omega$ is fixed, then in order that for every function $f\epsilon L^M$ we have

$$(*)\qquad \int_0^{2\pi} M(|f-s_n[f]|)dx\rightarrow 0$$

it is necessary that (⁑) $\liminf_{u\rightarrow\infty} M(2u)/M(u)>2$. Let $\varphi(t)$ be the right-hand derivative (which exists everywhere) of a given $M\epsilon\Omega$. If $\varphi'(t)$ exists and is nondecreasing and if $\varphi'(t)\rightarrow 0$ as $t\rightarrow\infty$, then condition (⁑) is also sufficient to ensure (*) for every $f\epsilon L^M$.

A. Zygmund.

Nikolsky, S. Approximation of functions in the mean by trigonometrical polynomials. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 207–256 (1946). (Russian. English summary)

Given a class $\mathfrak{M}$ of functions $f(x)$ of period 2π and a class of trigonometric polynomials $U_n(x, f)$ of order n, we may consider the best approximation

$$E_n(\mathfrak{M})=\sup_{f\epsilon\mathfrak{M}}\max_x|f(x)-U_n(x,f)|.$$

Usually,

$$(*)\qquad U_n(x,f)=\tfrac{1}{2}a_0+\sum_{k=1}^n\lambda_k(a_k\cos kx+b_k\sin kx),$$

where a_k, b_k are the Fourier coefficients of f, and $\lambda_1, \cdots, \lambda_n$ are prescribed numbers. As special cases we get here the partial sums of the Fourier series of f and their arithmetic means. For recent results concerning the behavior of $E_n(\mathfrak{M})$, see Kolmogoroff [Ann. of Math. (2) 36, 521–526 (1935)], Favard [Bull. Sci. Math. (2) 61, 209–224, 243–256 (1937)], Akhiezer and Krein [C. R. (Doklady) Akad. Sci. URSS (N.S.) 15, 107–111 (1937)] and Nikolsky [Trav. Inst. Math. Stekloff 15 (1945); these Rev. 7, 435]. In the present paper the author considers the parallel case when, in the definition of $E_n(\mathfrak{M})$, instead of $\|f-U_n\|_M=\max_x|f-U_n|$ we introduce the L distance

$$\|f-U_n(f)\|_L=\int_0^{2\pi}|f(x)-U_n(x,f)|dx.$$

The following theorems, selected out of many, indicate the character of the results. (1) Let W^rM and W^rL denote, respectively, the classes of periodic functions $f(x)$ such that the derivative $f^{(r-1)}$ exists, is absolutely continuous and its norm in the space of bounded or integrable functions does not exceed 1. Then, for any numbers λ_k, if $U_n(f)$ is defined by (*), we have $E_n(W^rL)\leqq E_n(W^rM)$. (2) If $S_n(x, f)$ is the partial sum of the Fourier series of f,

$$\sup_{f\epsilon W^rL}\|f-S_n\|_L=4\pi^{-2}n^{-r}\log n+O(n^{-r}),$$

an equation established for the metric M by Kolmogoroff [loc. cit.]. (3) If σ_n are the Fejér sums of f, then

$$E_{\sigma_n}(W^rL)-E_{\sigma_n}(W^rM)=O(n^{-r})$$

[see also Nikolsky, loc. cit., where the asymptotic value of $E_{\sigma_n}(W^rM)$ is found]. (4) Let $K(t)$ be periodic, integrable, and let $f(x)=\pi^{-1}\int_0^{2\pi}K(t-x)\varphi(t)dt$. A number of results are given connecting the best approximations of the functions f and φ, one in the space M, the other in L. Basic for the proofs are results of independent interest concerning Banach spaces.

A. Zygmund (Philadelphia, Pa.).

Nikolsky, S. La série de Fourier d'une fonction dont le module de continuité est donné. C. R. (Doklady) Acad. Sci. URSS (N.S.) 52, 191–194 (1946).

Let H_ω be the class of functions of period 2π having modulus of continuity $\omega(\delta)$, so that $|f(t')-f(t'')|\leqq\omega(|t'-t''|)$ for all t', t''. Let $S_n(x; f)$ be the nth partial sum of the Fourier series of f and let

$$E_n=E_{S_n}(H_\omega;x)=\sup_{f\epsilon H_\omega}|f(x)-S_n(x;f)|.$$

Generalizing a previous result [Trav. Inst. Math. Stekloff 15 (1945); these Rev. 7, 435] concerning functions of the class Lip α, the author shows that

$$E_n=2\pi^{-2}\theta_n\log n\int_0^{\pi/2}\omega(2z/\pi h)\sin z dz+O(\omega(h)),$$

where $\frac{1}{2}\leqq\theta_n\leqq 1$, $h=2\pi/(2n+1)$. If $\omega(\delta)$ is a convex function of δ, then $\theta_n=1$. A corresponding result is obtained for the case when the derivative $f^{(r)}$ exists and has modulus of continuity $\omega(\delta)$.

A. Zygmund (Philadelphia, Pa.).

Sansone, G. Sulla sommabilità delle serie trigonometriche di Fourier. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 45–48 (1946).

Let $S_n(x)$ be the partial sums of the Fourier series of a periodic and L-integrable $f(x)$. Let $p_n=bn+c$, where $b>0$ and c are integers. The author shows that $n^{-1}\sum_1^n S_{p_k}(x)\rightarrow f(x)$ for almost every x. [This is also contained in the theorem of Marcinkiewicz [J. London Math. Soc. 14, 162–168 (1939); these Rev. 1, 11] asserting that $N^{-1}\sum_1^N|S_n(x)-f(x)|\rightarrow 0$ for almost every x.]

A. Zygmund (Philadelphia, Pa.).

Cheng, Min-Teh. On strong summability of Fourier series. Acad. Sinica Science Record 1, 349–350 (1945).

The result previously established for k positive and even [same Record 1, 91–97 (1942); these Rev. 4, 272] is now extended to general positive k.

A. Zygmund.

Wang, Fu Traing. Some results of summability of a Fourier series. Acad. Sinica Science Record 1, 306–307 (1945).

Statements without proofs of results, some of which have meanwhile been published by the author elsewhere [cf.

Bull. Amer. Math. Soc. **50**, 412–416 (1944); Anais Acad. Brasil. Ci. **16**, 149–152 (1944); Duke Math. J. **12**, 77–87 (1945); these Rev. **5**, 237; **6**, 48, 172]. *A. Zygmund.*

v. Sz. Nagy, Béla. Approximation der Funktionen durch die arithmetischen Mittel ihrer Fourierschen Reihen. Acta Univ. Szeged. Sect. Sci. Math. **11**, 71–84 (1946).

A typical result of this paper is an explicit expression for the greatest deviation of an absolutely continuous function $f(x)$ from the nth arithmetic mean of its Fourier series, when $f(x)$ is of period 2π and (*) $|f'(x)|\leqq 1$ for all x. The corresponding problem is solved for higher Cesàro means of integral order δ and for the class of r-times differentiable functions ($r>1$, an integer), such that (**) $|f^{(r)}(x)|\leqq 1$; furthermore, for the class of functions where (**) is replaced by $|\tilde{f}^{(r)}(x)|\leqq 1$ and $\tilde{f}(x)$ is the function conjugate to $f(x)$. Thus some known results are sharpened and generalized. [Note of the reviewer: doubtless the assumption (*) can be replaced by the more general assumption that the difference-quotient of $f(x)$ is bounded by 1; furthermore, similar local theorems could be established. For recent literature on a related subject see A. Zygmund, Bull. Amer. Math. Soc. **51**, 274–278 (1945); Duke Math. J. **12**, 47–76 (1945); these Rev. **6**, 265; **7**, 60.] *O. Szász* (Cincinnati, Ohio).

Chen, Kien-Kwong. Functions of bounded variation and the Cesaro means of a Fourier series. Acad. Sinica Science Record **1**, 283–289 (1945).

Let $\varphi(t)\sim\sum A_n \cos nt$ and let σ_n^α be the (C, α) means for $\sum A_n$. If $\alpha t^{-\alpha}\int_0^t(t-u)^{\alpha-1}\varphi(u)du$ is of bounded variation in $(0, \pi)$, then $\sigma_n^\alpha-\sigma_{n-1}^\alpha=O(1/n)$ for $\alpha>0$. For $\alpha=0$ the result reduces to the familiar fact that the Fourier coefficients of functions of bounded variation are $O(1/n)$.

A. Zygmund (Philadelphia, Pa.).

Chen, Kien-Kwong. The super-absolute Cesaro summability of Fourier series. Acad. Sinica Science Record **1**, 290–299 (1945).

Let $\varphi(x)\sim\sum_1^\infty a_n \cos nx$. If $\sum|a_n| \log n<\infty$, the function $x^{-1}\int_0^x\varphi(t)dt$ is of bounded variation over $(0, \pi)$. Extension to the case of fractional means. *A. Zygmund.*

Chen, Kien-Kwong. The absolute convergence of the allied series of a Fourier series. Duke Math. J. **13**, 133–160 (1946).

Given a function $f(x)$ of period 2π, let us fix x and set $\varphi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}$, $\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\}$. In his previous paper [Amer. J. Math. **67**, 285–299 (1945); these Rev. **6**, 264; **7**, 620] the author showed that, if the functions $\varphi(t)$ and $t\varphi'(t)$ are of bounded variation over $0\leqq t\leqq\pi$, the Fourier series of f is absolutely convergent at the point x. The result is now extended to conjugate series: if $\psi(t)$ and $t\psi'(t)$ are of bounded variation in $(0, \pi)$, if the integral $\int_0^\pi t^{-1}|\psi(t)|dt$ is finite and if $\psi(\pi-0)=0$, then the series conjugate to the Fourier series of f converges absolutely at the point x. Various extensions are given.

A. Zygmund (Philadelphia, Pa.).

Sahai, Basdeo. On the summability of the conjugate series of the derived Fourier series. Proc. Nat. Acad. Sci. India. Sect. A. **10**, 93–102 (1940).

The author obtains some criteria for the Cesàro summability of the differentiated conjugate series of a Fourier series. His conditions, arising out of results of K. I. Sayers and S. Verblunsky, are expressed in terms of the function

$$\theta(t)=(2/\pi)\int_t^\infty u^{-2}\varphi(u)du,$$

where $\varphi(u)=\frac{1}{2}\{f(x+u)+f(x-u)\}$. The results would be more general if $\varphi(u)$ were replaced by $\varphi(u)-\alpha$, for a suitable α, so that a function like $\varphi(u)=\cos u$ might be included. The results are consistent with the fact that a criterion for the summability $(C, \lambda+1)$ of the differentiated conjugate series may be obtained by taking a criterion for the summability (C, λ) for the conjugate series and replacing $\psi(u)$ by $\{\varphi(u)-\alpha\}/u$, where $\psi(u)=\frac{1}{2}\{f(x+u)-f(x-u)\}$. [Cf. L. S. Bosanquet, Proc. London Math. Soc. (2) **49**, 63–76 (1945); these Rev. **7**, 154.] *L. S. Bosanquet* (London).

Minakshisundaram, S. Notes on Fourier expansion. I. J. London Math. Soc. **20**, 148–153 (1945).

The author proves the following theorem. For any k, let $\omega_n=\sin n_1x_1 \cdots \sin n_kx_k$, $\mu_n=n_1^2+\cdots+n_k^2$, $a_n=a_{n_1\cdots n_k}$. If $\sum\mu_n^{1k}a_n^2<\infty$, then a necessary and sufficient condition that the series $\sum a_n\omega_n(x)$ may converge by spherical partial sums at a given point x is that the spherical mean limit $f_1(x)$ of the sum function $f(x)$ shall exist at the point.

S. Bochner (Cambridge, Mass.).

Banerji, D. P. On the application of integral equation to the expansion of an arbitrary function in a series of special functions. Proc. Nat. Acad. Sci. India. Sect. A. **10**, 85–88 (1940).

The author's summary is as follows. In this paper I have selected two special functions $J_n^2(x)$ and $J_0^2(nt)$ which may be expressed as integral equations

$$J_n^2(x)=(2/\pi)\int_0^{\pi/2} J_{2n}(2x \sin \theta)d\theta,$$

$$J_0^2(nt)=(2/\pi)\int_0^{\pi/2} J_0(2nt \sin \theta)d\theta.$$

Expressing $F(x)$ as the integral equation

$$F(x)=(2/\pi)\int_0^{\pi/2} \varphi(x \sin \theta)d\theta$$

and expanding $\varphi(x \sin \theta)$ in a series of $J_{2n}(2x \sin \theta)$ or $J_0(2x \sin \theta)$, we get the expansion of $F(x)$ in a series of $J_n^2(x)$ or $J_0^2(nx)$. *H. Pollard* (Ithaca, N. Y.).

Olsson, Herbert. Expansions in series of Bessel functions of the second kind. II. Ark. Mat. Astr. Fys. **33A**, no. 12, 12 pp. (1946).

An earlier work [same Ark. **24B**, no. 5 (1934)] discussed convergence of Bessel function series. Now numerous consequences are derived. One sample: a necessary and sufficient condition that an integral function $\psi(z)$ have the Bessel function expansion $\psi(z)=\sum_0^\infty(-1)^{n+1}A_nI_n(z)$, with $\limsup\{(n-1)!|A_n|\}^{1/n}<\infty$, is that constants C_1, C_2 exist such that $\lim_{z\to\infty} z^{\frac{1}{2}}\psi(z)/(C_1e^z\pm C_2e^{-z})=1$, where the plus sign is to be used when $-\pi/2<\arg z<3\pi/2$, and the minus sign when $-3\pi/2<\arg z<\pi/2$. *I. M. Sheffer.*

Sidon, S. Über orthogonale Entwicklungen. Acta Univ. Szeged. Sect. Sci. Math. **10**, 206–253 (1943). [MF 16748]

This is a posthumous paper edited by G. Grünwald and P. Turán. It consists of four parts which can be read independently of each other. There are also two appendices containing miscellaneous results.

Part I contains extensions of the author's classical theorem to the effect that, if $\sum(a_\nu \cos n_\nu x + b_\nu \sin n_\nu x)$, $n_{\nu+1}/n_\nu > q > 1$, is the Fourier series of an integrable function bounded on one side, then $\sum(|a_\nu|+|b_\nu|)<\infty$. These extensions are not easy to state because of their specialized character.

The principal result of part II is the following. If $\{\varphi_n(x)\}$ is an orthonormal system on (a, b) such that $0<m<|\varphi_n(x)|<M$ for $n=1, 2, \cdots$ and $a \leqq x \leqq b$, then there exists a sequence of indices $n_1<n_2<\cdots$ such that for every null sequence ϵ_k an integrable function $f(x)$ can be found such that

$$\int_a^b f(x)\varphi_{n_k}(x)dx = \epsilon_k, \qquad k=1, 2, \cdots.$$

[For a similar result see J. Marcinkiewicz, Studia Math. 8, 1–27 (1939).]

The results of part III are typified by the following theorem. Let $\{n_k\}$ be a B_l sequence of integers (that is, the number of solutions of $n_{i_1}+n_{i_2}+\cdots+n_{i_l}=N$ is bounded by a number l independent of N), let

$$f(x)\epsilon L_s, \qquad s=lq/(lq-1),\ q>2,$$

and let $f(x)\sim\sum(a_\nu \cos \nu x + b_\nu \sin \nu x)$. Then

$$\sum(|a_{n_k}|^q+|b_{n_k}|^q)<\infty.$$

Part IV contains several theorems concerning Walsh series. As an example we may mention that there exists a Walsh series $\sum_0^\infty c_\nu\psi_\nu(x)$ such that $\limsup c_\nu>0$ and $\int_0^1|\sum_0^n c_\nu\psi_\nu(x)|dx=O(1)$. It should also be mentioned that this paper rectifies various omissions and minor errors in some of the author's previous papers. *M. Kac.*

Hilding, Sven H. On the closure of disturbed complete orthonormal sets in Hilbert space. Ark. Mat. Astr. Fys. 32B, no. 7, 3 pp. (1946).

Let $\{h_n\}$ be a complete orthonormal set in Hilbert space. The author considers the question of when a set $\{g_n\}$ sufficiently "close" to $\{h_n\}$ is also complete. The results are these. Let $r_n=\|g_n-h_n\|$. Then $\{g_n\}$ is also complete if (i) $\sum r_n^2<1$ or (ii) $\|g_n\|=1$ and $\sum r_n^2(1-\frac{1}{4}r_n^2)<1$, or (iii) $(g_n, h_n)=1$ and $\sum r_n^2/(1+r_n^2)<1$. The reviewer does not follow the remark on page 3 that the condition $\sum r_n^2<\infty$ is necessary in all three cases. [An extensive literature on the subject is ignored. A bibliography can be found in a paper of the reviewer, Ann. of Math. (2) 45, 738–739 (1944); these Rev. 6, 127]. *H. Pollard* (Ithaca, N. Y.).

Grünwald, G. On a theorem of S. Bernstein. Acta Univ. Szeged. Sect. Sci. Math. 10, 185–187 (1943). [MF 16745]

It is well known [G. Grünwald, Ann. of Math. (2) 37, 908–918 (1936); J. Marcinkiewicz, same Acta 8, 131–135 (1937)] that there exist continuous functions for which the Lagrange interpolation polynomials which are equal to them at the Chebyshev abscissas diverge everywhere. S. Bernstein [Comm. Soc. Math. Kharkoff (4) 5, 49–57 (1932)] has, however, shown that, if $\lambda>0$ is an arbitrarily small but fixed positive number and if we form the Lagrange polynomial of degree n which is equal to $f(x)$ at $(1-\lambda)n$ of the Chebyshev abscissas but can take any values we please at the remaining λn points, then the λn points and the values of the polynomials at these points can be so chosen that the sequence of polynomials thus obtained converges uniformly to $f(x)$.

Grünwald shows that Bernstein's result is best possible in the sense that the theorem becomes false if we replace the fixed λ by a sequence $\lambda_n\to0$. Corresponding to any sequence $\lambda_n\to0$ he constructs a continuous function $f(x)$ such that any sequence of Lagrange interpolation polynomials which are equal to $f(x)$ at $(1-\lambda_n)n$ Chebyshev abscissas will not be uniformly convergent whatever may be their values at the remaining $\lambda_n n$ points.

[There are a few misprints of no consequence. In equation (10) on p. 187 for $\leqslant$ read $\geqslant$ and in the following line for $\leqslant$ read $\geqslant$; for $<$ read $>$, and for c read $2c$.] *A. C. Offord* (Newcastle-upon-Tyne).

Feldheim, E. Una modificazione della formula di interpolazione di Hermite. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 77, 516–525 (1942). [MF 16258]

The author sets up a modification of Hermite's interpolation formula whereby a polynomial of degree at most $n+s-1$ is determined by its values at n distinct points $x_1, \cdots, x_n$ and by the values of its first derivative at s $(s<n)$ of these points. The case of $s=n-1$, where there is just one point, x_{m_n} say, at which the slope remains unspecified, receives special attention. Let $f(x)$ be continuous in $-1\leqq x\leqq1$. Let $P(x)$ be the polynomial of degree at most $2n-2$ interpolating $f(x)$ at the zeros of $\cos\{n(\arccos x)\}$ and having arbitrary but uniformly bounded slopes at the points x_ν $(\nu\neq m_n)$. It is shown that, unlike the case of the ordinary Hermite formula, the polynomials $P(x)$ need not converge uniformly to $f(x)$ as $n\to\infty$. However, it is shown that this convergence does hold uniformly in every closed interval $I\subset[a, b]$ which does not contain any limit points of the sequence $\{x_{m_n}\}$ of exceptional fundamental points. *I. J. Schoenberg* (Philadelphia, Pa.).

***Følner, Erling. Bidrag til de generaliserede næstenperiodiske Funktioners Teori. [Contribution to the Theory of Generalized Almost Periodic Functions].** Thesis, University of Copenhagen, 1944, 129 pp. (Danish)

Les espaces S^p, W^p ou B^p (notations familières à tous ceux qui s'occupent de ces questions) seront désignés par G^p $(p\geqq1)$; par G^p-p.p. on designe le sous-espace des fonctions presque-périodiques correspondantes. Un G^p-point (ou un G^p-p.p. point) de l'espace consiste dans l'ensemble des fonctions, appartenant à cet espace, et dont la distance G^p est nulle. Une fonction G^1-nulle qui est aussi G^{p_0} $(p_0>1)$ est G^p-nulle pour $p<p_0$. En un G^1-point (ou un G^1-p.p. point) il peut y avoir des G^p-fonctions (ou des G^p-p.p. fonctions); la borne supérieure P des p correspondants est appelée la longévité du G^1-point (ou du G^1-p.p. point) dans les espaces G^p (ou G^p-p.p.), $1\leqq P\leqq+\infty$; si $P<+\infty$, l'existence, ou la non-existence, d'un G^p-point (ou d'un G^p-p.p. point) est appelée le comportement au moment de la mort (vie ou mort au temps P). Un G^1-point peut avoir une longévité arbitraire et un comportement quelconque au moment de sa mort dans les G^p. Un G^1-point contenant un G^1-p.p. point a la même longévité dans les G^p et les G^p-p.p., mais les comportements au moment de la mort peuvent être différents si $G=S$ ou W car, si $G=B$, un B^1-p.p. vivant au temps p dans B^p est aussi vivant dans B^p-p.p.; néanmoins dans les B^p-p.p. le comportement au moment de la mort peut être quelconque. Il y a des fonctions S^1-p.p. de longévité P, vivantes dans S^P, mais mortes dans S^P-p.p. $(P>1)$. Dans un G^1-point $(G=W$ ou $B)$, non G-p.p., il y a des fonctions de longévité P_1 arbitraire, avec $1\leqq P_1\leqq P$, et de comportement arbitraire au temps de leur mort $(1<P_1<P)$. En un G^1-p.p. point $(G=W$ ou $B)$ le résultat est le même, avec ce complément qu'au moment P_1 de la mort, le comportement peut être arbitraire par rapport aux G^p et aux G^p-p.p., en égard du fait qu'une G^p-p.p. fonction est aussi une G^p-fonction. Les cas $P_1=1$ et $P_1=P$ sont aussi examinés et

donnent lieu à des résultats analogues. Les exemples donnés, instructifs quant aux divers modes de convergence, en marquent très heureusement les différences.

J. Favard (Paris).

★Bohr, Harald. An example of the application of the calculus of probability as an aid in mathematical analysis. Festskrift til Professor, Dr. Phil. J. F. Steffensen fra Kolleger og Elever paa hans 70 Aars Fødselsdag 28. Februar 1943, pp. 29–33. Den Danske Aktuarforening, Copenhagen, 1943. (Danish)

Using concepts familiar from the theory of probability the author gives a new proof of the following theorem. If $\{\lambda_n\}$ denotes a sequence of linearly independent numbers and $\sum A_n e^{i\lambda_n t}$ represents the Fourier series of an almost periodic function, then $\sum |A_n|$ is convergent.

František Wolf (Berkeley, Calif.).

★Petersen, Richard. On the Laplace transformation of an almost periodic function. Festskrift til Professor, Dr. Phil. J. F. Steffensen fra Kolleger og Elever paa hans 70 Aars Fødselsdag 28. Februar 1943, pp. 133–139. Den Danske Aktuarforening, Copenhagen, 1943. (Danish)

The author proves the following theorem. The Laplace transform F of an almost periodic f, with characteristic exponents $\{\lambda_n\}$ not everywhere dense, is analytic in the whole plane except at the points $\{i\lambda_n\}$. At an isolated point $i\lambda_n$, F has a simple pole. The proof uses approximations of f by finite trigonometric polynomials.

František Wolf.

Kober, H. On components of a function and on Fourier transforms. Amer. J. Math. **68**, 398–416 (1946).

It is well known [cf. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford University Press, 1937, chap. V] that if $F(x)$ is such that its square is integrable over $(-\infty, \infty)$ then it can be expressed in the form $F(x)=F_1(x)+F_2(x)$, where $F_1(x)$ and $F_2(x)$ are the limits when $|y|\rightarrow 0$ of functions $F_1(z)$ and $F_2(z)$, $z=x+iy$, which are regular in the half-planes $y>0$ and $y<0$, respectively. The author deals with the corresponding problem when $F(x)$ is such that $\int_{-\infty}^{\infty}|F(x)|(1+x^2)^{-1}dx<\infty$. Using his results he shows that, if

$$g_k(t)=\int_{-\infty}^{\infty}(1-itx/k)^{-k-1}F(x)dx$$

is such that $\int_{-\infty}^{\infty}|g_k(t)|dt\leqq M$ uniformly in k, then $F(x)$ is a Fourier-Stieltjes transform.

A. C. Offord.

Levin, B. J. On a generalization of the Fejer-Riesz theorem. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 291–294 (1946).

The theorem referred to in the title is that a nonnegative trigonometric polynomial can be written as the square of the absolute value of a trigonometric polynomial. The author uses a theorem of Wiener and Pitt [Duke Math. J. **4**, 420–436 (1938)] to generalize this to functions $f(x)=\int_{-A}^{A}e^{ixt}d\sigma(t)$ in the following way. Let $\sigma(t)$ be of bounded variation, $\sigma(t)=\sigma_a+\sigma_s+\sigma_d$, where the subscripts indicate the absolutely continuous, singular and discontinuous components; let $f(x)\geqq c>0$ in $(-\infty, \infty)$; let

$$\inf_x\left|\int_{-A}^{A}e^{ixt}d\sigma_d(t)\right|>\int_{-A}^{A}|d\sigma_s(t)|;$$

then

$$f(x)=\left|\int_{-\frac{1}{2}A}^{\frac{1}{2}A}e^{ixt}d\tau(t)\right|^2,$$

where $\tau(t)$ is of bounded variation. As an application, the author proves the following theorem of A. Artemenko [unpublished thesis]. Any positive definite function on $(-A, A)$ can be extended so that it is positive definite on $(-\infty, \infty)$. The case of a continuous function was considered by M. Krein [same C. R. (N.S.) **26**, 17–22 (1940); these Rev. **2**, 361].

R. P. Boas, Jr. (Providence, R. I.).

Shanker, Hari. On functions which are Fourier sine or cosine-transforms of each other. Proc. Nat. Acad. Sci. India. Sect. A. **11**, 73–77 (1941).

The functions

$$\varphi(x)=(2\pi)^{-\frac{1}{2}}\Gamma(m+1)D_n(x)\{e^{i\mu\pi}D_{-m-1}(ix)+e^{-i\mu\pi}D_{-m-1}(-ix)\},$$

$$\psi(x)=n!\exp(-\tfrac{1}{2}x^2)x^{m-n}L_n^{m-n}(x^2)$$

are Fourier sine or cosine transforms of each other according as $\mu=\frac{1}{2}(n+1)$ or $\frac{1}{2}n$, where n is a positive integer and m is any number greater than $n-1$.

M. C. Gray.

Gupta, H. C. Two self-reciprocal functions. Proc. Nat. Acad. Sci. India. Sect. A. **13**, 37–39 (1943).

The author obtains two R_ν functions in the form of finite series of generalized hypergeometric functions of type ${}_3F_2$.

M. C. Gray (New York, N. Y.).

Mital, P. C. On self-reciprocal functions. Proc. Nat. Acad. Sci. India. Sect. A. **13**, 42–43 (1943).

Using a known theorem [B. M. Mehrotra, Proc. London Math. Soc. (2) **34**, 231–240 (1932)] that if $f(x)$ is R_ν then $g(p)=p\int_0^{\infty}(y^2+p^2)^{-1}f(y)dy$ is R_ν, the author derives some R_ν functions from known R_ν's.

M. C. Gray.

★Potier, Robert, et Laplume, Jacques. Le calcul symbolique et quelques applications à la physique et à l'électricité. Actualités Sci. Ind., no. 947. Hermann et Cie., Paris, 1943. 148 pp.

This text, which opens with the statement, "le calcul symbolique est très mal connu en France," is designed to make available in French material which is commonplace in other languages. It contains essentially the material to be found in standard American texts such as that of Carson [Electrical Circuit Theory, McGraw-Hill, New York, 1926] and, as in that book, the particular calculus presented is based on the unilateral Laplace transform. This limits the range of applicability to initial-condition problems. There are an extensive table and many illustrative examples of the traditional variety.

H. Pollard (Ithaca, N. Y.).

Lévy, Paul. Le calcul symbolique et ses principales applications. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 41–56 (1946).

Expository article.

Wagner, Karl Willy. Laplacesche Transformation und Operatorenrechnung. Arch. Electrotechnik **35**, 502–506 (1941).

A brief survey of the Laplace transform and its standard applications to physical problems.

H. Pollard.

Possenti, Renzo. Sulle relazioni fra le parti reali e le parti immaginarie degli operatori funzionali. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **4**, 454–459 (1943).

The author discusses the inversion of the Laplace transformation by means of an integration along the imaginary axis and the role that the real and the imaginary parts of $f(is)$ play in such inversion, $f(s)$ being the symbol of the

Laplace transform. The paper is connected with operational methods in the theory of electric circuits.
I. Opatowski (Ann Arbor, Mich.).

Shastri, N. A. Some theorems in operational calculus. Proc. Benares Math. Soc. (N.S.) 7, 3–9 (1945).

The author evaluates some Laplace transforms by means of the Whittaker functions. *H. Pollard.*

Parodi, Maurice. Sur la transformée de Laplace de la fonction $\delta(t)$ de Dirac. Revue Sci. (Rev. Rose Illus.) 82, 105 (1944).

The author observes that inconsistencies arise if the Dirac delta function is integrated like an ordinary function. He does not point out that the difficulties are avoided by Riemann-Stieltjes integration. *H. Pollard.*

Tewari, N. D. A theorem on the generalised Laplace's transform. Proc. Benares Math. Soc. (N.S.) 7, 51–58 (1945).

The author evaluates

$$(1)\qquad \phi_m^k(p)=p\int_0^\infty (2xp)^{-\frac{1}{2}}W_{k,m}(2xp)f(x)dx$$

for functions which satisfy the integral equation

$$(2)\qquad f(x)=\int_0^\infty (xy)^{\frac{1}{2}}J_\nu(xy)f(y)dy,$$

by substituting (2) into (1) and carrying out the integration with respect to x. He investigates a number of particular cases (special values of k and m) and examples (special functions f), thereby obtaining a number of identities.
A. Erdélyi (Edinburgh).

Churchill, Edmund. Information given by odd moments. Ann. Math. Statistics 17, 244–246 (1946).

L'aûteur démontre qu'il existe toujours une fonction de répartition admettant des moments d'ordre impair donnés quelconques, et en déduit que, quels que soient les nombres $\{\mu_{2n-1}\}$ $(n=1, 2, \cdots)$, la fonction de répartition $F(x)$, et ϵ, il existe une fonction de répartition $F^*(x)$ admettant les μ_{2n-1} comme moments d'ordre impair et telle que l'on ait, pour tout x, $|F(x)-F^*(x)|<\epsilon$. Autrement dit, la connaissance des moments d'ordre impair d'une fonction de répartition $F(x)$ n'apporte pas grande information sur elle, et en particulier l'emploi habituel du troisième moment comme mesure de l'asymétrie de $F(x)$ peut être abusif.
R. Fortet (Caen).

Verblunsky, S. Two moment problems for bounded functions. Proc. Cambridge Philos. Soc. 42, 189–196 (1946).

The paper is a contribution to the theory of moments if only a finite number of moments are involved. For instance, if numbers $s_0, s_1, \cdots, s_{n-1}$ and $t_0, t_1, \cdots, t_{n-1}$ are connected by $\exp(\sum_0^{n-1}s_k/z^{k+1})=1+\sum_0^{n-1}t_k/z^{k+1}+\cdots$ then there will exist a function $f(x)$, where $0\leqq f\leqq 1$, such that $s_k=\int_{-\infty}^0 x^k f(x)dx$, $k=0, \cdots, n-1$, if and only if $t_k=\int_{-\infty}^0 x^k d\sigma$, $k=0, \cdots, n-1$, for $d\sigma\geqq 0$. For n odd, this is also true for the total line $(-\infty, \infty)$. *S. Bochner* (Cambridge, Mass.).

Polynomials, Polynomial Approximations

Verblunsky, S. On positive polynomials. J. London Math. Soc. 20, 73–79 (1945). [MF 16698]

The author establishes representations of a polynomial p_n which is positive in an interval. If $p_n\geqq 0$ for all x then $p_n=A^2+B^2$; if $p_n\geqq 0$ for $x\geqq 0$ then $p_n=A^2+xB^2$; if $p_n\geqq 0$ for $-1\leqq x\leqq 1$ then $p_n=A^2+(1-x^2)B^2$ in case n is even and $p_n=(1+x)A^2+(1-x)B^2$ in case n is odd. Here A and B are polynomials. In some cases the author sharpens old results and in other cases he gives new proofs of known theorems. Other theorems are also proved. *A. C. Schaeffer.*

Visser, C. A generalization of Tchebychef's inequality to polynomials in more than one variable. Nederl. Akad. Wetensch., Proc. 49, 455–456 = Indagationes Math. 8, 310–311 (1946). [MF 16828]

The original inequality states that if $P(u)=a_nu^n+\cdots$ is a polynomial of degree n then

$$\max_{-1\leqq u\leqq 1}|P(u)|\geqq 2^{-n+1}|a_n|.$$

The author proves that if $P(u_1, \cdots, u_h)$ is a polynomial of total degree n in h variables with real coefficients then

$$\max_{-1\leqq u_i\leqq 1}|P(u_1, \cdots, u_h)|\geqq 2^{-n+1}\max_{|u_i|=1}|P^*(u_1, \cdots, u_h)|,$$

where P^* denotes the sum of the terms of P whose precise degree is n. The inequality is exact. *A. C. Schaeffer.*

Ibraguimoff, I. I. Sur la valeur asymptotique de la meilleure approximation d'une fonction qui possède un point critique réel. C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 238–240 (1945).

Bernstein has given a method for finding the best approximation to the function $f(x)=(a-x)^s$, where $a>1$, by polynomials of degree n for $-1\leqq x\leqq 1$. The author applies Bernstein's methods to the function

$$f(x)=(a-x)^s[\log(a-x)]^m, \qquad a>1.$$

The corresponding problem for trigonometric polynomials was discussed by de la Vallée Poussin [Leçons sur l'Approximation des Fonctions, Paris, 1919, p. 143]. Let $E_n[f(x)]$ denote the best approximation for $f(x)$ on $-1\leqq x\leqq 1$ by a polynomial $P_n(x)$; the author gives asymptotic formulae for $E_n\{(a-x)^s[\log(a-x)]^m\}$ when m is any positive number and s a real nonintegral positive number and when s is a positive integer. He also gives inequalities for

$$E_n\{(1-x)^s[\log(1-x)]^m\}$$

for similar values of m and s. *A. C. Offord.*

Nikolsky, S. On the best approximation of functions satisfying Lipschitz's conditions by polynomials. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 295–322 (1946). (Russian. English summary)

It is well known [Favard, Bull. Sci. Math. (2) 61, 209–224 (1937); Akhyeser and Krein, C. R. (Doklady) Acad. Sci. URSS (N.S.) 15, 107–111 (1937)] that for any function $f(x)$ of period 2π satisfying the condition $(*)$ $|f(x'')-f(x')|\leqq M|x''-x'|$ the best approximation by trigonometric polynomials of order $n-1$ is at most $M\pi/(2n)$, and that the result is best possible. The author investigates the corresponding problem of the best approximation $E_n(f)$ of functions f defined in the interval $(-1, 1)$ and satisfying condition $(*)$. He obtains that $E_n(f)\leqq M\pi/(2n)-\epsilon_n$, where $\epsilon_n>0$, $\epsilon_n=O(1/n\log n)$, and that the result is the best possible. Clearly, for any fixed $f(x)$, $-1\leqq x\leqq 1$, satisfying $(*)$, we have $\lim nE_n(f)\leqq M\pi/2$. It is shown that for some f we actually have the equality sign here. Let $U_n(f, x)=\frac{1}{2}a_0+\sum_1^{n-1}a_kP_k(x)$ be the $(n-1)$th partial sum of the Fourier series of a function $f(x)$, $-1\leqq x\leqq 1$, with respect to the orthogonal system of functions $P_k(x)=\cos k\arccos x$.

It is shown that

$$\sup_f |f(x)-U_n(f,x)| = 4M\pi^{-2}n^{-1}\log n\,(1-x^2)^{\frac{1}{2}}+O(n^{-1})$$

uniformly in x, the upper bound on the left being taken with respect to all functions f satisfying (*). Similarly, if $\eta_k=(k\pi/2n)\cot(k\pi/2n)$, $V_n=\frac{1}{2}a_0+\sum_1^{n-1}a_k\eta_kP_k$, then

$$\sup_f |f(x)-V_n(f,x)| \leqq M\pi(2n)^{-1}(1-x^2)^{\frac{1}{2}}+|x|O(n^{-2}\log n).$$

A. Zygmund (Philadelphia, Pa.).

Frank, Evelyn. The location of the zeros of polynomials with complex coefficients. Bull. Amer. Math. Soc. **52**, 890–898 (1946).

To determine the number of zeros of a given nth degree polynomial in a circle C: $|z|=r$, the author maps the interior of C upon the half-plane $\Re(w)>0$ by means of the transformation $z=r(w-1)/(w+1)$ and then applies to the polynomial $P_r(w)=(w+1)^nP(z)$ her previous results [same Bull. **52**, 144–157 (1946); these Rev. **7**, 295] on the number of zeros in the half-plane $\Re(z)>0$. This involves the expansion of the ratio

$$\{P_r(w)+(-1)^n\bar{P}_r(-w)\}/\{P_r(w)-(-1)^n\bar{P}_r(-w)\}$$

in one of the forms $F(w)$ or $1/(k_0+F(w))$, where $k_0\neq 0$ and where

$$F(w)=\frac{1}{c_1w+k_1+}\,\frac{1}{c_2w+k_2+}\cdots\frac{1}{+c_nw+k_n},\qquad c_p\neq 0.$$

If in either form k of the c_p are negative and the remaining $n-k$ of the c_p are positive, then $P(z)$ has k zeros inside and $n-k$ zeros outside the circle C. On the basis of this result, an algorithm is set up for the computation of the number of zeros of $P(z)$ in an annular ring $r_1<|z|<r_2$.

M. Marden (Milwaukee, Wis.).

Leonhard, A. Neues Verfahren zur Stabilitätsuntersuchung. Arch. Elektrotechnik **38**, 17–28 (1944).

Let $F(p)$ be the steady-state transmission ratio around the feedback loop of a servomechanism. Generally, $F(p)=B(p)/A(p)$, where $A(p)$ and $B(p)$ are polynomials in p with the degree of $A(p)$ greater than that of $B(p)$. For the servomechanism to be stable, it is necessary and sufficient that $F(p)\neq -1$ when $\Re(p)>0$, that is, (I) $H(p)=A(p)+B(p)\neq 0$ for $\Re(p)>0$. Writing $H(i\omega)=u(\omega)+iv(\omega)$ with ω, $u(\omega)$ and $v(\omega)$ real variables, the author proves that condition (I) will be satisfied if and only if all the zeros of $u(\omega)$ and $v(\omega)$ are real, with the zeros of $u(\omega)$ separating those of $v(\omega)$. He applies this theorem to some examples of servomechanisms, with the purpose of showing that the theorem is simpler to use than the Nyquist or Hurwitz criteria. [Reviewer's note. The theorem is essentially the same as one proved by Biehler, J. Reine Angew. Math. **87**, 350–352 (1879), and Hermite, Bull. Soc. Math. France **7**, 128–131 (1879).]

M. Marden (Milwaukee, Wis.).

Biernacki, M. Sur les zéros des polynomes et sur les fonctions entières dont le développement taylorien présente des lacunes. Bull. Sci. Math. (2) **69**, 197–203 (1945).

By use of Grace's theorem, the author derives a new bound on the least number $\phi(n,p)$ such that, if a polynomial $P(x)$ of degree n has $n-p$ zeros in a circle of radius R, its first derivative $P'(x)$ will have at least $n-p-1$ zeros in the concentric circle of radius $R\phi(n,p)$. The new bound is $\phi(n,p)\leqq(n+p)!(n-p+1)!/n!(n-1)!$, a result which is for small p better than, but for large p not as good as, the bound $\phi(n,p)\leqq\csc\pi/2(p+1)$ given by M. Marden [Trans. Amer. Math. Soc. **45**, 355–368 (1939)]. The new bound leads to the following generalization of Montel's theorem [Ann. Sci. École Norm. Sup. (3) **40**, 1–34 (1923)]. If all the zeros of $P(x)=a_0+a_1x+\cdots+a_px^p$ lie in the circle $|x|\leqq R$, then all the zeros of $f_k(x)=P(x)+a_{n_1}x^{n_1}+\cdots+a_{n_k}x^{n_k}$ lie in the circle

$$|x|\leqq R_k=\prod_{i=1}^{p}\frac{n_i}{n_i-p}\,\frac{n_i+p-1}{n_i-p+1}\,\frac{n_i+p-2}{n_i-p+2}\cdots\frac{n_i+1}{n_i-1}R.$$

As $k\to\infty$, the entire function $f(x)=\lim f_k(x)$ consequently takes on every value infinitely often, provided the series $\sum(1/n_j)$ converges. *M. Marden* (Milwaukee, Wis.).

Gaspar, Fernando L. On a property of real numbers. Publ. Inst. Mat. Univ. Nac. Litoral **6**, 329–340 (1946). (Spanish) [MF 16935]

Let a_i, $i=1,2,\cdots,n$, be an arbitrary set of n real numbers. Define $m_s=\sum_{i=1}^n a_i^s$. Let $p,q,r,s,\cdots,n$ be any set of positive integers. By using the results of a previous paper [Revista Unión Mat. Argentina **8**, 81–90 (1942); these Rev. **4**, 196], the author constructs a set of polynomials $\{p_n(x)\}$ orthogonal over the set of points a_i, that is, such that,

$$\sum_{i=1}^n p_j(a_i)p_k(a_i)\begin{cases}=0, & j\neq k,\ j=k\geqq n;\\ \neq 0, & j=k<n.\end{cases}$$

These polynomials have the form

$$p_0(x)=x^p,\quad p_1(x)=\frac{-1}{\Delta_1}\begin{vmatrix}x^p, & x^q\\ m_{2p}, & m_{p+q}\end{vmatrix},$$

$$p_2(x)=\frac{(-1)^2}{\Delta_2}\begin{vmatrix}x^p, & x^q, & x^r\\ m_{2p}, & m_{p+q}, & m_{p+r}\\ m_{p+q}, & m_{2q}, & m_{q+r}\end{vmatrix},$$

$$p_3(x)=\frac{(-1)^3}{\Delta_3}\begin{vmatrix}x^p, & x^q, & x^r, & x^s\\ m_{2p}, & m_{p+q}, & m_{p+r}, & m_{p+s}\\ m_{p+q}, & m_{2q}, & m_{q+r}, & m_{q+s}\\ m_{p+r}, & m_{q+r}, & m_{2r}, & m_{r+s}\end{vmatrix},\cdots;$$

Δ_i, $i=1,\cdots,n$, is the complementary minor of x^t, the last element of the first row of each determinant.

In a former paper [Fac. Ci. Mat. Fis. Nat. Publ. Téc.-Ci. Rosario, no. **10** (1937), in particular, p. 33] the author has shown that $\Delta_i\geqq 0$. It is from this fact that a chain of inequalities may be deduced which imply the property of an arbitrary set of real numbers a_i which the author wishes to point out. Thus, it is shown that for all sets (a_i) and $(p,q,r,s,\cdots,n)$ as defined, we have (1) $m_{2p}m_{2q}\geqq m_{p+q}^2$,

$$(2)\quad m_{2p}m_{2q}m_{2r}+2m_{p+q}m_{q+r}m_{p+r}\geqq m_{p+r}^2m_{2q}+m_{p+q}^2m_{2r}+m_{q+r}^2m_{2p},$$

and so forth.

The author extends these results by considering systems of polynomials in two variables orthogonal over a set of points in the plane and indicates briefly the analogous results in the case where the domain over which a set of functions is orthogonal is a region D of n-dimensional space.

M. A. Basoco (Lincoln, Neb.).

Ionesco, D. V. Relations entre polynomes définis par certaines relations de récurrence. Mathematica, Timişoara **22**, 102–108 (1946).

The nth iterate z_n of the linear fractional transformation $z_1=(az+b)/(cz+d)$ has the form $z_n=(P_nz+Q_n)/(R_nz+S_n)$. If $ad-bc=1$, then $U_n=P_n+S_n$ is a real polynomial of degree n in the quantity $x=a+d$, with numerical coefficients. Numerous properties of these polynomials are established, some of them relating U_n to the Gegenbauer polynomials $V_n(x)$ (defined by $(1-tx+t^2)^{-1}=\sum V_n(x)t^n$). Some typical

formulas: (i) $U_n=xU_{n-1}-U_{n-2}$, (ii) $U_n=V_n-V_{n-2}$, (iii) $(x^2-4)U_n''+xU_n'-n^2U_n=0$. *I. M. Sheffer.*

Steffensen, J. F. On the polynomials $R_\nu^{[\lambda]}(x)$, $N_\nu^{[\lambda]}(x)$ and $M_\nu^{[\lambda]}(x)$. Acta Math. 78, 291–314 (1946).

The polynomials of the title relate to, and are extensions of, the "poweroids" considered earlier by the author [Acta Math. 73, 333–366 (1941); these Rev. 3, 236]. Numerous properties are established. Let $\theta=\varphi(D)=\sum_{\nu=1}^{\infty}k_\nu D^\nu$, $k_1\neq 0$, be a differential operator (D^ν the νth derivative operator) and let $x^{\overline{\nu}|}$ be the corresponding set of "poweroids." Then, λ being constant,

$$R_\nu^{[\lambda]}(x)=(D/\theta)^\lambda x^{\overline{\nu}|},\quad (t/\varphi(t))^\lambda e^{xt}=\sum_{\nu=0}^{\infty}t^\nu R_\nu^{[\lambda]}(x)/\nu!;$$

if θ_1 is a second such operator, which may be written $\theta_1=\varphi_1(\theta)=\sum_{\nu=1}^{\infty}h_\nu\theta^\nu$, $h_1\neq 0$, then

$$N_\nu^{[\lambda]}(x)=(\theta/\theta_1)^\lambda x^{\overline{\nu}|},\quad (\zeta/\varphi_1(\zeta))^\lambda e^{xt}=\sum_{\nu=0}^{\infty}\zeta^\nu N_\nu^{[\lambda]}(x)/\nu!,\quad \zeta=\varphi(t).$$

The definition of $M_\nu^{[\lambda]}(x)$ may be omitted here. The author did not observe that (for λ fixed) the polynomials $R_\nu^{[\lambda]}(x)$ are Appell polynomials and the polynomials $N_\nu^{[\lambda]}(x)$ and $M_\nu^{[\lambda]}(x)$ belong essentially to the class designated as "type zero" by the reviewer [Duke Math. J. 5, 590–622 (1939); these Rev. 1, 15]. *I. M. Sheffer* (State College, Pa.).

Jackson, Dunham. The boundedness of orthonormal polynomials on certain curves of the third degree. Bull. Amer. Math. Soc. 52, 899–907 (1946).

Further results on the subject of the title. [For earlier work and references, cf. Trans. Amer. Math. Soc. 58, 167–183 (1945); these Rev. 7, 64.] The curves now treated are $y^2=x^2(x+1)$, $y^2=x^2(x-1)$, $y^2=x^3-x$, $y^2=x^3+x$; except for modifications of details, the method is that of the earlier papers. In treating numerous particular cases the author evidently had in mind the hope that the insight so acquired may yield the basis for a general theory. *I. M. Sheffer.*

Bula, Clotilde A. On some polynomials in two variables analogous to those of Laguerre. Publ. Inst. Mat. Univ. Nac. Litoral 6, 305–314 (1946). (Spanish) [MF 16932]

The author deals with the problem of determining a system of polynomials $L_{r,s}(x,y)$ in two variables which may be considered as a logical extension of the system of Laguerre polynomials $L_n(x)$, whose characteristic property is the normalized orthogonality relation

$$\int_0^\infty e^{-x}L_n(x)L_m(x)dx=\delta_{n,m}.$$

It is found that, if the polynomials $L_{r,s}(x,y)$ are arranged so that they appear in the order defined by the sequence $L_{n-j,j}(x,y)$, $n=0,1,2,\cdots$; $j=0,1,2,\cdots,n$, then the desired system may be determined so as to satisfy the condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x^2+y^2)^{\frac12}}L_{p,q}(x,y)L_{m,n}(x,y)dydx\begin{cases}=0, & (p,q)\neq(m,n),\\ \neq 0, & (p,q)=(m,n).\end{cases}$$

The polynomials $L_{p,q}(x,y)$ are normalized in the sense that the coefficient of x^py^q is unity. The first ten polynomials of the sequence are given explicitly.

The solution depends on a paper by F. L. Gaspar [Fac. Ci. Mat. Fis. Nat. Publ. Téc.-Ci., Rosario, no. 15 (1938), in particular, p. 30]. *M. A. Basoco* (Lincoln, Neb.).

Special Functions

Rutgers, J. G. Some identities. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde 52, 163–167. (1943). (Dutch. German, English and French summaries) [MF 15776]

From the results of an earlier paper [Nederl. Akad. Wetensch., Proc. 45, 929–936, 987–993 (1942); these Rev. 6, 64] the author deduces a number of elementary identities involving binomial coefficients and gamma functions. The simplest of them [already obtained in the paper cited] is

$$2^{2r}\sum\binom{2k}{2p_1+2r}\binom{2p_1+2r-1}{2p_2+2r-1}\cdots\binom{2p_{2r-1}+1}{2p_{2r}+1}=\sum_{p=0}^{r}\frac{(-)^p(2r-2p+1)^{2k+1}}{p!(2r-p+1)!}.$$

On the left side the summation extends over all nonnegative integer values of $p_1,\cdots,p_{2r}$ for which the binomial coefficients do not vanish; k is a nonnegative integer. *A. Erdélyi* (Edinburgh).

Shabde, N. G. On some results involving Legendre functions. Proc. Benares Math. Soc. (N.S.) 7, 1–2 (1945).

Evaluation of

$$\int_{-1}^{1}(1-x^2)^{-\frac12 mr}P^m_{n_1+m}(x)\cdots P^m_{n_r+m}(x)P_{n_1+\cdots+n_r}(x)dx$$

and results for two other integrals. *A. Erdélyi.*

Jaeger, J. C. On the repeated integrals of Bessel functions. Quart. Appl. Math. 4, 302–305 (1946).

In an important class of problems of the theory of electricity, the Laplace transforms of the solutions contain powers of $\{1+(p^2+1)^{\frac12}\}$ or $\{p+1+(p^2+1)^{\frac12}\}$ in the denominator. The object of this note is to show that the "original" functions can be expressed in terms of repeated integrals of Bessel functions. For the r-fold integral of $nJ_n(t)/t$ from 0 to t the author writes $n\mathrm{Ji}_n^{(r)}(t)\,(n>0)$ and $t^{r-1}/(r-1)!\,(n=0)$, and for that of $J_n(t)$, $\mathrm{Ji}_{n,r}(t)$ $(n\geqq 0)$. He obtains, for example,

$$L\{J_n(t)+3\mathrm{Ji}_{n,2}(t)+2\mathrm{Ji}_{n,4}(t)-2n\mathrm{Ji}_n^{(2)}(t)-2n\mathrm{Ji}_n^{(4)}(t)\}=\frac{(p^2+1)^{\frac12}\{(p^2+1)^{\frac12}-p\}^n}{\{1+(p^2+1)^{\frac12}\}^2},\quad n\geqq 0,$$

and

$$L\{J_{n+2}(t)-2J_{n+1}(t)+J_n(t)+\mathrm{Ji}_{n+2,2}(t)-2\mathrm{Ji}_{n+1,2}(t)+\mathrm{Ji}_{n,2}(t)\}=\frac{4(p^2+1)^{\frac12}\{(p^2+1)^{\frac12}-p\}^n}{\{p+1+(p^2+1)^{\frac12}\}^2}.$$

As an example of the way in which the above functions arise, a semi-infinite artificial transmission line with mid-series termination is considered, in which the series elements are inductances and the shunt elements are condensers. *S. C. van Veen* (Delft).

Meijer, C. S. Multiplikationstheoreme für die Funktion $G^{m,n}_{p,q}(z)$. Nederl. Akad. Wetensch., Proc. 44, 1062–1070 (1941). [MF 15767]

In this and several other papers [cf. the four following reviews] the author investigates the generalised hypergeometric function

$$G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\prod_{\mu=1}^{m}\Gamma(b_\mu-s)\prod_{\nu=1}^{n}\Gamma(1-a_\nu+s)}{\prod_{\mu=m+1}^{q}\Gamma(1-b_\mu+s)\prod_{\nu=n+1}^{p}\Gamma(a_\nu-s)}z^s ds,$$

where $q\geqq 1$, $0\leqq n\leqq p\leqq q$, $0\leqq m\leqq q$. The integral can be evaluated, by means of the residue calculus, as a finite combination of hypergeometric functions ${}_pF_{q-1}$ and this representation provides an alternative definition [adopted in this paper] of the G-function. The main result of the present paper is a group of four multiplication theorems for the G-function. The first multiplication theorem is

$$\lambda^{-b_1}G(\lambda w\,|\,b_1)=\sum_{r=0}^{\infty}(1-\lambda)^r G(w\,|\,r+b_1)/r!,$$

where all parameters of the G-function which are the same on both sides have been omitted. The other three multiplication theorems similarly express the expansion of $\lambda^{-b_q}G(\lambda w)$ in powers of $1-\lambda$ and the expansion of $\lambda^{1-a_1}G(\lambda w)$ and of $\lambda^{1-a_p}G(\lambda w)$ in powers of $1-1/\lambda$. Special values of the parameters give multiplication theorems for Whittaker functions, Struve functions and Lommel functions. *A. Erdélyi.*

Meijer, C. S. On the *G*-function. I. Nederl. Akad. Wetensch., Proc. **49**, 227–237 = Indagationes Math. **8**, 124–134 (1946). [MF 16577]

This is the first instalment of a longer paper and contains a summary of Barnes's results on the asymptotic expansion of the G-function for large z and the analytic continuation of this function outside the unit circle in simple cases, namely, for $p=q$, $m+n\geqq p+1$ and $m=p=q$, $n=1$. The main purpose of the investigation is stated in the second instalment [cf. the following review]. *A. Erdélyi.*

Meijer, C. S. On the *G*-function. II. Nederl. Akad. Wetensch., Proc. **49**, 344–356 = Indagationes Math. **8**, 213–225 (1946). [MF 16583]

[Cf. the preceding review.] The G-function satisfies the differential equation

$$\left\{(-)^{p-m-n}z\prod_{j=1}^{p}\left(z\frac{d}{dz}-a_j+1\right)-\prod_{j=1}^{q}\left(z\frac{d}{dz}-b_j\right)\right\}y=0.$$

The fundamental solutions of this equation valid in the vicinity of the origin are generalised hypergeometric series ${}_pF_{q-1}$; the expression of G in terms of these solutions is the one obtained from the Barnes integral by means of the calculus of residues. The author now proceeds to obtain a fundamental system of solutions valid near infinity and poses the question of expressing $G(z)$ as a linear combination of these solutions; the most important part of the present series of papers centers round this problem.

The work is based on a considerable number of definitions and lemmas which it is impossible to describe in detail here. The lemmas of the present part concern, apart from identities between symbols introduced in the definitions, relations between $G^{m,n}_{p,q}(z)$ and $G^{m+1,n}_{p,q}(ze^{\pm\pi i})$ or $G^{m,n+1}_{p,q}(ze^{\pm\pi i})$ and other relations of a similar character. *A. Erdélyi.*

Meijer, C. S. On the *G*-function. III. Nederl. Akad. Wetensch., Proc. **49**, 457–469 = Indagationes Math. **8**, 312–324 (1946). [MF 16829]

[Cf. the preceding review.] After more preliminary considerations the author obtains the first of four expansion formulae "which render it possible to write the function $G^{m,n}_{p,q}$ in a special way as a linear combination of functions $G^{k,l}_{p,q}$ and $G^{k,l-1}_{p,q}$." "These expansion formulae are the most powerful instruments of the present paper." The first expansion represents $G^{m,n}_{p,q}$ as a combination of $n-l+1$ functions of the type $G^{k,l}_{p,q}$, $1\leqq l\leqq n$, $m\leqq k\leqq q$, $m+n\geqq k+l$.
A. Erdélyi (Edinburgh).

Meijer, C. S. On the *G*-function. IV. Nederl. Akad. Wetensch., Proc. **49**, 632–641 = Indagationes Math. **8**, 391–400 (1946).

The second expansion formula represents $G^{m,n}_{p,q}$ in terms of $n-l+1$ functions of the type $G^{k,l}_{p,q}$ and r functions of the type $G^{k,l-1}_{p,q}$; $0\leqq l-1\leqq n$, $m\leqq k\leqq q$, $r\geqq \max(0, k+l-m-n)$. The third expansion formula represents $G^{m,n}_{p,q}$ in terms of $n-l+1$ functions $G^{k,l}_{p,q}$, r functions of the type $G^{k,l-1}_{p,q}$ and $k+l-m-n-r$ other functions of the same type; $0\leqq l-1\leqq n$, $m\leqq k\leqq q$, r is an arbitrary integer.

This part contains some more lemmas, presumably in preparation for the fourth expansion. *A. Erdélyi.*

McLachlan, N. W. Mathieu functions and their classification. J. Math. Phys. Mass. Inst. Tech. **25**, 209–240 (1946).

The author considers solutions of Mathieu's differential equation and of the associated equation arising from Mathieu's if a purely imaginary independent variable is introduced. The solutions in question are described as Mathieu functions and are associated with pairs of parameter values such that either one or both solutions of Mathieu's equation are periodic, almost periodic, or nonperiodic functions of the independent variable. The objects of the paper are (a) to extend the representations of such solutions previously published, (b) to define second solutions of the equations in each case, (c) to introduce a number of new forms of solutions of integral and of fractional order, (d) to classify the various solutions. No derivations are given of the formulas believed to be new. Some observations are made as to the convergence of the series involved. The bulk of the paper refers to integral periodic and associated nonperiodic solutions of both Mathieu equations, including infinite Fourier, Bessel, Neumann and Hankel series, as well as integral equations and representations, totalling several hundred individual expressions. Relationships between the solutions are given. Then fractional periodic and nonperiodic (pseudoperiodic) solutions are considered, including means of computation of the characteristic exponent and of the parameter values. Finally, asymptotic expressions for the functions are considered, including a table of dominant terms. Real zeros of some solutions in the case of large positive independent variables are given. The classification aims at showing the usefulness of the notations introduced.
M. J. O. Strutt (Eindhoven).

Campbell, Robert. Fonctions spéciales. Recherche d'équations intégrales et de la valeur asymptote des fonctions de Mathieu associées. C. R. Acad. Sci. Paris **222**, 1069–1071 (1946).

Campbell, Robert. Sur une généralisation des fonctions de Mathieu normées. C. R. Acad. Sci. Paris **222**, 269–271 (1946).

Campbell, Robert. Sur les solutions de période $2s\pi$ de l'équation de Mathieu associée. C. R. Acad. Sci. Paris **223**, 123–125 (1946).

The author considers the solution of the differential equation arising in the separation of the wave equation in spheroidal coordinates, $(1-Z^2)U''-2ZU'+(a+k^2f^2Z^2)U=0$. He seems to have overlooked previous work which contains all of his results. [See, for example, Stratton, Morse, Chu and Hutner, Elliptic Cylinder and Spheroidal Wave Functions, Wiley, New York, 1941; these Rev. **4**, 281.] *H. Feshbach* (Cambridge, Mass.).

Functional Analysis, Ergodic Theory

Wielandt, Helmut. Das Iterationsverfahren bei nicht selbstadjungierten linearen Eigenwertaufgaben. Math. Z. **50**, 93–143 (1944). [MF 15843]

The first part of this paper is concerned with properties of principal solutions [Hauptlösungen] of a general linear operator on an abstract linear space, with special attention to a decomposition theorem involving an arbitrary orthogonalization process. The author then proves convergence properties of the sequence of iterates $\{\mathfrak{K}^{(\nu)}y\}$ of the linear operator $\mathfrak{K}y$ for the following cases: (i) algebraic operators, $\mathfrak{K}y=\sum_{\beta=1}^{n}a_{\alpha\beta}y_\beta$, $\alpha=1, \cdots, n$; (ii) integral operators, $\mathfrak{K}y=\int_0^1K(x, \xi)y(\xi)d\xi$, $0\leqq x\leqq 1$, and (iii) mixed integral operators of the form $\mathfrak{K}y=\int_0^1K(x, \xi)y(\xi)d\xi+\sum_{\beta=1}^{n}k_\beta(x)y(x_\beta)$, $0\leqq x\leqq 1$. The more explicit results obtainable when all the considered characteristic values of the operator are of order one and have distinct absolute values are amplified for two-point boundary problems of the form

$$dy/dx+A(x)y=\lambda B(x)y,\quad R_0y(0)+R_1y(1)=\lambda[S_0y(0)+S_1y(1)],$$

where $A(x)$, $B(x)$ are $n\times n$ matrices of continuous functions on $0\leqq x\leqq 1$, and R_0, R_1, S_0, S_1 are constant $n\times n$ matrices, together with corresponding boundary problems involving a single nth order linear differential equation. For the case of algebraic operators involving 2×2 matrices conditions are determined for the sequence of iterates to converge "monotonically" with respect to a particular metric; in addition, an example involving 3×3 matrices is presented to illustrate the use of a transformation to increase the rapidity of convergence. *W. T. Reid* (Evanston, Ill.).

Liapounoff, A. Sur les fonctions-vecteurs complètement additives. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 277–279 (1946). (Russian. French summary)

The present paper is a continuation of an earlier one with the same title [same Bull. **4**, 465–478 (1940); these Rev. **2**, 315]. For definition of terms we refer to the review of that paper. In the earlier paper it was proved that the set of values of any completely additive vector-function with values in an n-dimensional Euclidean space is convex. In the present paper it is shown that this theorem is no longer true if the value-space is infinite-dimensional, even if compact. This is shown by an example in which the value-space is a compact parallelepiped in l_1. *J. V. Wehausen.*

Lewitan, B. M. A generalization of the operation of translation and infinite hypercomplex systems. III. Rec. Math. [Mat. Sbornik] N.S. **17(59)**, 163–192 (1945). (English. Russian summary) [MF 16667]

This chapter concludes a study begun earlier [same Rec. N.S. **16(58)**, 259–280; **17(59)**, 9–44 (1945); these Rev. **7**, 254]. Where chapter II considers operators generated by cubic matrices, the present one provides an extension to the continuous case. This requires the prior development of a theory of the equation $u''+\rho(t)u=0$ for infinite intervals, which is accomplished by use of results of Weyl [Math. Ann. **68**, 220–269 (1910)]. The end results are analogous to those of the preceding chapter. *H. Pollard.*

Catunda, Omar. Sui sistemi di equazioni alle variazioni totali in più funzionali incogniti. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **12**, 751–764 (1941).

The paper appeared in Portuguese in Anais Acad. Brasil. Ci. **14**, 109–125 (1942); these Rev. **5**, 6.

Titov, N. S. On different kinds of convergence of elements and linear operators in Banach spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 569–572 (1946).

Let E and E' be Banach spaces. A sequence of elements $x_1, x_2, \cdots$ of E is defined to be (E')-convergent to $x\epsilon E$ whenever for each bounded linear operator u mapping E into E' the sequence $u(x_1)$, $u(x_2)$, $\cdots$ converges to $u(x)$ with respect to the norm in E'. If $E'=E$ then (E')-convergence coincides with strong convergence; if every bounded linear operator from E into E' is totally continuous, in particular, if E' is finite dimensional, then (E')-convergence coincides with weak convergence. In general, strong convergence implies (E')-convergence and (E')-convergence implies weak convergence. Examples are given showing that (E')-convergence need not coincide with either strong or weak convergence and that (E')-convergence may coincide with weak convergence even when E and E' are infinite dimensional.

In the second half of the paper (E'')-convergence for bounded linear operators from E into E' is defined and discussed similarly. The paper concludes with a proof of the fact that the unit sphere in the space of bounded linear operators from E into E' is sequentially compact with respect to (E'')-convergence provided that E and the conjugate of E' are separable and that E'' is one dimensional. This result has as consequences two known theorems on the weak compactness of unit spheres. *G. W. Mackey.*

Bourgin, D. G. Approximate isometries. Bull. Amer. Math. Soc. **52**, 704–714 (1946).

Continuant les recherches entreprises par Hyers et Ulam [Bull. Amer. Math. Soc. **51**, 288–292 (1945); ces Rev. **7**, 123] l'auteur étudie les "isométries approchées" des espaces de Banach; une isométrie approchée est une transformation T d'un espace de Banach E_1 dans un espace de Banach E_2 telle que $|\|T(y)-T(x)\|-\|y-x\||\leqq\epsilon$ pour x et y quelconques dans E_1 (ϵ fixe). Il montre que si E_2 est un espace uniformément convexe satisfaisant à deux conditions supplémentaires (vérifiées par les espaces L^p pour $1<p<\infty$), $U(x)=\lim_{n\to\infty}T(2^nx)/2^n$ est une isométrie de E_1 dans E_2; si T applique E_1 sur E_2, il en est de même de U et si en outre $T(0)=0$, on a $\|T(x)-U(x)\|\leqq 12\epsilon$ dans E_1. Hyers et Ulam avaient démontré ces résultats [loc. cit.] lorsque E_1 et E_2 sont identiques à un espace de Hilbert.

J. Dieudonné (São Paulo).

Sigalov, A. G. Représentations presque isométriques et la pseudo-dérivabilité. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 11–12 (1946).

Let E be a normed linear space. A function f defined for $0<t\leqq 1$ with values in E is called pseudo-derivable at 0 if (a) $\lim_{t\to 0}\|f(t)\|/t$ exists and (b) whenever two sequences $\{t_k\}$, $\{t_k'\}\to 0$ are so chosen that $t_k=O(t_k')$ and $t_k'=O(t_k)$, then $\|f(t_k)/t_k-f(t_k')/t_k'\|\to 0$ as $k\to\infty$. A transformation T of a neighborhood U of the zero element of E to a neighborhood V of the zero of another normed linear space is called almost isometric if $T(0)=0$ and for every $\epsilon>0$ there exists $\delta>0$ such that $|x|$ and $|x'|<\delta$ imply $|\|x-x'\|-\|Tx-Tx'\||<\epsilon(\|x\|+\|x'\|)$. The author proves that, if T is almost isometric, if f with values in U is pseudo-derivable at 0, and if $\varphi(t)=Tf(t)$ for $0<t\leqq 1$, then φ is also pseudo-derivable.

M. M. Day (Urbana, Ill.).

Grunblum, M. M. Concerning my note on "biorthogonal systems in Banach space." C. R. (Doklady) Acad. Sci. URSS (N.S.) 52, 387 (1946).

The paper appeared in the same C. R. 47, 75–78 (1945) [these Rev. 7, 125]. The author acknowledges that some of his results were previously obtained by Krein, Milman and Rutman in a paper which was unavailable to him [Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 16, 106–110 (1940); these Rev. 3, 49].

Zaanen, A. C. On a certain class of Banach spaces. Ann. of Math. (2) 47, 654–666 (1946).

The class to which the title refers is a generalization of the class L_p of Lebesgue spaces. The space L_Φ is a function space in which the requirement is made that for any element x of the space the function $\Phi[|x(t)|]$ is integrable, where $\Phi(u)=\int_0^u\varphi(\bar{u})d\bar{u}$ and $\varphi(\bar{u})$ is nondecreasing for $\bar{u}\geqq 0$, $\varphi(0)=0$ and $\lim_{\bar{u}\to\infty}\varphi(\bar{u})=\infty$. If $\psi(\bar{v})$ is the inverse to $\varphi(\bar{u})$, $\Psi(v)=\int_0^v\psi(\bar{v})d\bar{v}$ and the functions $\Phi(u)$ and $\Psi(v)$ play the same role as conjugate exponents in Lebesgue spaces. The Banach spaces L_Φ^* are the spaces of functions $\{x(t)\}$ such that $\int_a^b x(t)y(t)dt$ exists for all y in L_Ψ with the norm defined as $\|x\|_\Phi=\sup|\int_a^b x(t)y(t)dt|$ for y such that $\int_a^b\Psi(|y|)dt\leqq 1$. For the theorems proved in the paper the assumption is always made that there exists a constant $C>0$ such that $\Phi(2u)\leqq C\Phi(u)$. The author proves that the space L_Φ^* is separable and that the most general linear functional over L_Φ^* is given by $F(y)=\int_a^b y(t)f(t)dt$, where $f\varepsilon L_\Psi^*$ with the norm satisfying the inequalities $\frac{1}{2}\|f\|_\Psi\leqq\|F\|\leqq\|f\|_\Psi$. In the latter part of the paper it is shown that the transformation $Ky=\int_a^b K(s,t)y(t)dt$, where $K(s,t)$ is measurable over $a\leqq s,t\leqq b$, $K(s,t)\varepsilon L_\Psi^*$ for almost all s and $k(s)=\|K(s,t)\|_\Psi\varepsilon L_\Phi^*$ is a completely continuous transformation in L_Φ^* as is also its adjoint. This fact makes possible the application of Fredholm's theorems for equations of the type $\int_a^b K(s,t)dt-\lambda y(s)=z(s)$ and its adjoint equation, where y, $z\varepsilon L_\Phi^*$ and $K(s,t)$ is of the type described above. The methods used are of the same type as those used for the spaces L_p. *R. E. Fullerton* (Madison, Wis.).

Rothe, E. H. Gradient mappings in Hilbert space. Ann. of Math. (2) 47, 580–592 (1946).

The author considers mappings of the form (1) $y=x+F(x)$ defined in a sphere $\|x\|\leqq r$ of a real Hilbert space E. If F is completely continuous (1) is called a mapping with completely continuous displacement; if the range of F is contained in a finite-dimensional subspace, (1) is called a layer mapping. If $i(x)$ is a real-valued scalar function in the sphere, the mapping $y=f(x)$ is called the gradient of i, $f=\text{grad } i$, if $i(x_2)-i(x_1)=(f(x_1),x_2-x_1)+R$, where $R/\|x_2-x_1\|\to 0$ as $x_2\to x_1$. If certain additional conditions are satisfied f is called the strict gradient of i. Reasons are given for regarding a gradient mapping as the natural generalisation to the nonlinear case of a symmetric linear operator.

A scalar function $s(x)=\frac{1}{2}(x,x)+S(x)$ is called a layer scalar if there is a finite-dimensional subspace E^m such that $S(x)$ takes the same value for all x having the same projection on E^m. The author shows that if $f=\text{grad } i$ is continuous, then f is a layer mapping if and only if i is a layer scalar. A definition of completely continuous scalar functions is given and it is shown that if the gradient of a completely continuous scalar is continuous it must be completely continuous. A partial converse of this result is obtained by defining strictly completely continuous mappings; if F is strictly completely continuous and is the strict gradient of I, then I is completely continuous. *F. Smithies.*

Martchenko, V. Sur les fonctions dont les distances à certains ensembles dans l'espace des fonctions bornées sont égales. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 663–666 (1946).

Let M be the ring of all bounded real-valued functions defined on a set A; let $N\subset M$ be a multiplicative semigroup containing the unit function. For $f_1,\cdots,f_n\varepsilon N$ and $\epsilon>0$ let S be the set of all pairs (x,y) of elements of A satisfying $|f_1(x)-f_1(y)|<\epsilon,\cdots,|f_n(x)-f_n(y)|<\epsilon$. For any $g\varepsilon M$, define

$$S[g]=\sup_{z\varepsilon A}\ \sup_{(x,y),(x,z)\varepsilon S}|f(y)-f(z)|;$$

and let $[g]$ be the infimum of all $S[g]$. The theorem presented here is that $[g]=2\inf\|g-f\|$ for $f\varepsilon L$, where L is the linear hull of N and $\|h\|=\sup|h(x)|$ for $x\varepsilon A$ when $h\varepsilon M$. It is pointed out that this result implies M. H. Stone's generalization of Weierstrass' theorem of approximation by polynomials. The proof of the present theorem utilizes a generalization of the polynomials of S. N. Bernstein. *R. Arens* (Princeton, N. J.).

Schwartz, Laurent. Sur les fonctions moyenne-périodiques. C. R. Acad. Sci. Paris 223, 68–70 (1946).

If E is a topological vector space of complex functions of the real variable x, the author calls a function $f(x)$ "mean-periodic" if the closure of all finite linear combinations of translations of $f(x)$ is not the whole space. Mean-periodic subspaces are similarly defined. The author announces a number of properties of mean-periodic subspaces and functions and, in particular, asserts that the latter are identical with the mean-periodic functions of Delsarte [J. Math. Pures Appl. (9) 14, 403–453 (1935)] if E is the space of continuous functions with a topology based on uniform convergence on every finite interval. *R. H. Cameron.*

Fantappiè, Luigi. Nuovi fondamenti della teoria dei funzionali analitici. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 12, 617–706 (1942).

The purpose of this paper is to extend some of the author's results on analytic functionals of analytic functions of a single variable [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (6) 3, 453–683 (1930); Jber. Deutsch. Math. Verein. 43, 1–25 (1933)] to analytic functions of n variables. For this purpose he considers the space $S^{(n)}$ of all functions $y(z_1,\cdots,z_n)$ locally analytic and bi-regular in their entire field of definition. A function is bi-regular at a finite point if it is developable in a power series around that point, and bi-regular at an infinite point if there exists a homogeneous transformation of the homogeneous coordinates of $z_0,z_1,\cdots,z_n$ of the vicinity of the infinite point into a vicinity of a finite point in which the transformed function is regular, and where the function vanishes at every point corresponding to an infinite point of the original vicinity. A topology is set up in this space, the vicinity (A,σ) of a function y_0, which is bi-regular in a region B_0 containing the closed set A, being the totality of all functions y, defined and bi-regular in a region containing A, and such that $|y-y_0|<\sigma$ on A. Such a space is a T_0 space. If R is a linear region of $S^{(n)}$, i.e., an open set closed under addition and multiplication by complex coefficients, then there exists a closed set A such that R consists of all functions of $S^{(n)}$ defined and bi-regular in A. The author considers functionals defined on a region R of S analytic in the sense that if $y(z,\alpha)$ are functions of R analytic in α then $F(y(z,\alpha))$ is analytic in α. In particular, linear analytic functionals defined on a linear region R are considered. It is proved that, if $v(\alpha)$ is the indicatrix of

the linear analytic functional F, i.e.

$$v(\alpha)=F[1/(\alpha_1-z_1)\cdots(\alpha_n-z_n)],$$

then $F(y)$ is expressible in the form

$$\frac{1}{(2\pi i)^n}\int_{C_1}\cdots\int_{C_n}v(t_1,\cdots,t_n)y(t_1,\cdots,t_n)dt_1\cdots dt_n,$$

where the curves $C_1,\cdots,C_n$ are each closed curves enclosing the projections $A_1,\cdots,A_n$ of the closed set A on each z_i-plane, and the formula is valid for all y which are defined and bi-regular on some region B containing the topologic product of the curves $C_1,\cdots,C_n$ and their interiors.

T. H. Hildebrandt (Ann Arbor, Mich.).

Beurling, Arne. Sur quelques formes positives avec une application à la théorie ergodique. Acta Math. **78**, 319–334 (1946).

The author succeeds in proving, for all p in $1\leqq p\leqq 2$, that, in case of any one-parameter group U_t, $-\infty<t<\infty$, of linear isometries of any L^p-space into itself, the distance function $\rho(t)=\|U_tf-f\|$ admits of a representation

$$\rho(t)=\int_0^\infty(1-\cos\alpha t)d\mu(\alpha)$$

with nondecreasing $\mu(\alpha)$ for which $\int_0^\infty d\mu(\alpha)=2^{1/p}\|f\|$. In particular, for the L^p-space of periodic functions with $U_tf=f(s+t)$, one obtains numbers $A_n\geqq 0$ such that

$$\int_0^{2\pi}|f(s+t)-f(s)|^pds=\sum_1^\infty A_n(1-\cos nt).$$

In the latter case the author also shows that

$$\int_0^{2\pi}|f(s+t)-g(s)|^pds=\|f\|^p+\|g\|^p-\sum_{-\infty}^{\infty}A_n(f,g)e^{int},$$

where $\sum_{-\infty}^{\infty}|A_n(f,g)|\leqq 2(\|f\|\cdot\|g\|)^{p/2}$. An analogous relation holds for $(-\infty,\infty)$ instead of $0<x\leqq 2\pi$. *S. Bochner.*

Rickart, C. E. Banach algebras with an adjoint operation. Ann. of Math. (2) **47**, 528–550 (1946).

The algebras considered are normed rings (commutativity not assumed) with complex scalars. Such algebras are called B^*-algebras if to every x there exists an x^* such that (1) $(x^*)^*=x$, (2) $(xy)^*=y^*x^*$, (3) $(\lambda x+\mu y)^*=\bar\lambda x^*+\bar\mu y^*$, λ and μ being complex numbers; (4) $\|xx^*\|=\|x\|^2$. If in addition one assumes (5) for every x, x^*x+e possesses an inverse, then it is known that the algebra is isomorphic to an algebra of operators in Hilbert space [Gelfand and Neumark, Rec. Math. [Mat. Sbornik] N.S. 12(54), 197–213 (1943); these Rev. **5**, 147]. The question as to whether axiom 5 is independent of the others is still open. The author bases his development on axioms 1–4 only. Fundamental to his program is the theorem of Gelfand and Neumark that if the B^*-algebra is commutative with bicompact space of maximal ideals $\mathfrak{M}$, then B^* is isomorphic to the ring of all continuous complex functions on $\mathfrak{M}$.

The particular B^*-algebras considered are those having "sufficiently many" projections. These are called B_p^*-algebras. In an operator algebra, the requisite number of projections can be introduced by requiring that the algebra is closed in the weak topology. In the present instance, this is done by requiring the following. If $h\epsilon B_p^*$ is Hermitian $(h^*=h)$, then there is in B_p^* a projection u such that $hu=h$ and if, for some Hermitian k, $hk=0$ then $uk=0$. Thus, in an operator algebra, u is the projection on the range of h. It is shown that the projections of a commutative B_p^*-algebra form a σ-lattice; also that any B_p^*-algebra is generated by its projections.

An element x is relatively regular if there exists a z such that $xzx=x$. For these elements a unique relative inverse can be defined. If $x\epsilon B_p^*$ is normal $(xx^*=x^*x)$ then x is relatively regular if and only if $\lambda=0$ is either not in, or an isolated point of, the spectrum of x. The relatively regular elements are dense in any left or right ideal of a B_p^*-algebra.

The remainder of the paper is devoted to an abstract study of projections along lines initiated by Murray and von Neumann in their studies of rings of operators. An element p is partially isometric if $p^*p=\epsilon u$ with u a projection and $\epsilon=\pm 1$ (if axiom 5 is invoked, the possibility $\epsilon=-1$ does not arise). Two projections u and v are equivalent, $u\sim v$, if there exists a partially isometric p such that $p^*p=\epsilon u$, $pp^*=\epsilon v$. If there is a projection v' such that $v'v=v'$ and $u\sim v'$ then $u\precsim v$. If $v=v_1+\cdots+v_n$ with $v_iv_j=0$, $i\neq j$, and $v_i\precsim u$, then v is finite over u. A necessary and sufficient condition that a B_p^*-algebra is simple is that, for every pair of projections u, v with $u\neq 0$, v is finite over u. The notions are related to that of quasi-transitivity defined for an algebra $\mathfrak{A}$ as follows: $x\mathfrak{A}y=0$ implies $x=0$ or $y=0$. An algebra satisfies the denumerability condition if every class of mutually orthogonal projections is at most denumerable. Such a B_p^*-algebra is quasi-transitive if and only if it is central. *E. R. Lorch* (New York, N. Y.).

Gottschalk, W. H. Almost periodic points with respect to transformation semi-groups. Ann. of Math. (2) **47**, 762–766 (1946).

Let X be a regular topological space, let T be a topological space in which a multiplicative binary operation is defined and let f be a continuous transformation of $X\times T$ into X. If $x\epsilon X$ and $t\epsilon T$, the image of $x\times t$ under f is denoted by $f^t(x)$. It is assumed that f defines a transformation semi-group in the sense that $f^s(f^t(x))=f^{ts}(x)$. A point x of X is almost periodic if, corresponding to any neighborhood U of x, there exists a compact set A in T such that each left translate of A contains an element t for which $f^t(x)\epsilon U$. The set $f(x,T)$ is the orbit of x and its closure, denoted by $\Gamma(x)$, is called the orbit closure of x. As a first result it is shown that if x is almost periodic then $\Gamma(x)$ is a minimal orbit-closure; if $\Gamma(x)$ is compact, the converse is true. The relationship of pointwise almost periodicity and the partitioning of X by the orbit closures is developed. It is shown that, if X is locally compact and zero-dimensional, if T is the semi-group of positive integers and each point of X is contained in its orbit-closure, then f is pointwise almost periodic. This implies that if X is locally compact and zero-dimensional, pointwise recurrence (stability in the sense of Poisson) implies pointwise almost periodicity. The author shows that this implication does not follow from metrizability and compactness of X. In order to characterize the case in which the orbit closures form a continuous partition, a concept of weak almost periodicity of f is introduced and studied. There are a number of additional related results. *G. A. Hedlund* (Charlottesville, Va.).

Calculus of Variations

Deladrière, R. Sur la réduction des équations paramétriques dans le calcul des variations. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 103–109 (1946).

This note deals with the parametric problem with one independent variable t and n dependent variables $z^1,\cdots,z^n$. The integrand f is a function of z and its derivatives of the first m orders. The purpose of this paper is twofold: first, to

transform the Euler equations into the form $z^iQ_{ik}=0$, where $Q_{ik}=-Q_{ki}$ have special properties; second, to reduce these equations in a symmetric way to a system of $n-1$ equations.
M. R. Hestenes (Chicago, Ill.).

Faedo, Sandro. Ricerche sul comportamento asintotico delle soluzioni delle equazioni e dei sistemi di equazioni differenziali. Ann. Mat. Pura Appl. (4) **23**, 25–50 (1944). [MF 16610]

The first part of this paper is devoted to a résumé of results previously obtained by the author for the calculus of variations problem of minimizing an integral of the form

$$I_\infty=\int_a^\infty f\{x, y_1(x), \cdots, y_n(x), \cdots, y_1^{(m)}(x), \cdots, y_n^{(m)}(x)\}dx.$$

Some of these results, dealing with integrals I_∞ whose integrand functions are of the second degree in the mn variables $y_1, \cdots, y_n^{(m)}$, are then used to establish results on the asymptotic character of solutions of self-adjoint ordinary linear differential equations of even order, with particular attention to equations of the second and fourth orders. In addition, some results are given on the existence and uniqueness of extremals with certain asymptotic properties for an integral of the form

$$I_\infty=\int_a^\infty \{\theta^2(x)\Phi(y')+\phi^2(x)\psi^2(y)\}dx,$$

where $\theta^2(x)>0$, $\lim_{|y'|\to\infty}|\Phi^2(y')/y'|=+\infty$ and the integral $\int_a^\infty\{\theta^2(x)\Phi^2(0)+\phi^2(x)\psi^2(0)\}dx$ is convergent.
W. T. Reid (Evanston, Ill.).

Hestenes, Magnus R. The Weierstrass E-function in the calculus of variations. Trans. Amer. Math. Soc. **60**, 51–71 (1946).

Hestenes, Magnus R. Theorem of Lindeberg in the calculus of variations. Trans. Amer. Math. Soc. **60**, 72–92 (1946).

Hestenes, Magnus R. Sufficient conditions for the isoperimetric problem of Bolza in the calculus of variations. Trans. Amer. Math. Soc. **60**, 93–118 (1946).

These papers are concerned with an isoperimetric formulation of the problem of Bolza. It is desired to minimize a functional $I(C)=g(a)+\int_C f(a, y, \dot y)dt$ in a class of arcs a^h, $y^i(t)$, $t^1\leqq t\leqq t^2$; $h=1, \cdots, r$; $i=0, \cdots, n$, satisfying conditions $\varphi^\beta(a, y, \dot y)=0$, $\beta=1, \cdots, m<n$, $y^i(t^1)=T^{i1}(a)$, $y^i(t^2)=T^{i2}(a)$, $I^\sigma(C)=g^\sigma(a)+\int_C f^\sigma(a, y, \dot y)dt=0$, where the a^h are constants. It is assumed as usual that φ^β, f^σ and f are positively homogeneous in $\dot y^i$. In the first paper the author develops properties of the E-function in terms of E-dominance. This concept is defined as follows. Let $\mathfrak{D}$ be the set of admissible elements (a, y, p) satisfying $\varphi^\beta(a, y, p)=0$ and let C_0 be an arc whose elements $(a, y, \dot y)$ are in $\mathfrak{D}$. Then a function $F(a, y, p)$ is said to E-dominate another function $H(a, y, p)$ near C_0 if there is a neighborhood $\mathfrak{D}_0$ relative to $\mathfrak{D}$ of the elements $(a, y, \dot y)$ on C_0 and a constant $b>0$ such that $E_F(a, y, p, q)\geqq b|E_H(a, y, p, q)|$ holds for (a, y, p) in $\mathfrak{D}_0$ and (a, y, q) in $\mathfrak{D}$. The author shows that the integrand $(p^ip^i)^{\frac12}$ and the f^σ are E-dominated by $l^0f+l^\sigma f^\sigma+m^\beta(a, y)\varphi^\beta$ if and only if the arc C_0 is nonsingular and satisfies the strengthened Weierstrass condition II_N with the multipliers l^0, l^σ, $m^\beta[a, y(t)]$. The paper contains a number of other properties of E-dominance and of allied concepts such as weak dominance.

In the second paper the author obtains an analogue of the theorem of Lindeberg, that is, an extension of a similar theorem by Reid for the nonparametric case. With the help of this and allied results, he shows that the sufficiency theorems for isoperimetric problems can be obtained from those for nonisoperimetric ones and that the sufficiency theorems for parametric problems can be obtained from those for nonparametric ones. Furthermore, he shows that the sufficiency theorems for the problem of Bolza are obtainable from those for the problems of Mayer.

The last paper is devoted primarily to establishing a sufficiency theorem for strong relative minima. The theorem was conjectured by McShane and was proved by him for weak minima. The principal result states that under certain hypotheses on an arc C_0 there is a neighborhood of C_0 in (a, y)-space and a constant $\epsilon>0$ such that

$$I(C)-I(C_0)\geqq\min(\epsilon, \epsilon K(C, C_0))$$

for every admissible arc C in the neighborhood, where

$$K(C, C_0)=|a-a_0|^2+\max|y(t)-y_0(t)|^2+\int_0^1|\dot y(t)-\dot y_0(t)|^2dt.$$

The methods used are an extension of those used by McShane [same Trans. **52**, 344–379 (1942); these Rev. **4**, 48] and Myers [Duke Math. J. **10**, 73–97 (1943); these Rev. **4**, 200].
H. H. Goldstine (Princeton, N. J.).

Mancill, Julian D. Multiple integral problems of the calculus of variations with prescribed transversality coefficients. Actas Acad. Ci. Lima **8**, 155–164 (1945).

The author derives necessary and (if the region of definition possesses suitably simple connectivity properties) sufficient conditions in order that three functions

$$T_i(x, y, z; A, B, C)$$

should be such that the equation $T_1w_x+T_2w_y+T_3w_z=0$ is a transversality condition for the problem of minimizing a parametric double integral with a nonvanishing integrand.
J. E. Wilkins, Jr. (Buffalo, N. Y.).

Mathematical Statistics

Haden, H. G. A note on the distribution of the different orderings of n objects. Proc. Cambridge Philos. Soc. **43**, 1–9 (1947).

The author shows that, for large n, the distribution of the number of permutations of n elements with s inversions, defined as the number of interchanges to effect the natural ordering $1, 2, \cdots, n$, suitably normalized, is normal with mean $n(n-1)/4$ and variance $n(n-1)(2n+5)/72$. This has been proved previously by H. B. Mann [Econometrica **13**, 245–259 (1945); these Rev. **7**, 21] with greater brevity by the method of moments.
J. Riordan.

Cheng, Tseng-Tung. A simplified formula for mean difference. Coll. Papers Sci. Engin. Nat. Univ. Amoy **1**, 69–72 (1943).

The paper also appeared in J. Amer. Statist. Assoc. **39**, 240–242 (1944); these Rev. **6**, 91.
J. W. Tukey.

Pailloux, H. Sur un problème de répartition. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 123–125 (1946).

Heuristic remarks on an operational solution of the integral equation

$$\pi^{-\frac12}\int_{-\infty}^\infty e^{-(y-x)^2}f(y)dy=\phi(x).$$

The left side represents the effect of the superposition of

Gaussian errors of measurement with actual deviations with the unknown density function $f(x)$; the function $\phi(x)$ can be observed and should be considered known.

W. Feller (Ithaca, N. Y.).

von Schelling, H. Bemerkungen zur Verteilung von Pascal. Naturwissenschaften 29, 517–518 (1941).

The author states that his remarks were prompted by the fact that while the Pascal distribution is considered well known in Scandinavia it appears to be unknown in Germany. Known relations are mentioned and the problem of estimating the parameter is touched upon. *W. Feller.*

Jones, Howard L. Linear regression functions with neglected variables. J. Amer. Statist. Assoc. 41, 356–369 (1946).

In making forecasts on the basis of empirical regression functions fitted by the method of least squares, forecasting errors too large to be reasonably attributable to chance alone may arise, for example, because (1) the errors in the dependent variable may be serially correlated or (2) significant variables may have been neglected. This paper considers the mean square error and other features of the linear regression function exposed to the second difficulty alone. The mean square error is analyzed into a sampling variance independent of neglected variables and a mean square bias having a minimum for the proper choice of the linear function. Two theorems and six corollaries (given as explicit formulas) are obtained. The theorems and corollaries are discussed and implications explored in connection with Student's t and Snedecor's F and in view of conflicting objectives it is proposed, among other suggestions, that unless $F>2$ one or more variables might appropriately be dropped. *A. A. Bennett* (Providence, R. I.).

Bhargava, R. P. Test of significance for intra-class correlation when family sizes are not equal. Sankhyā 7, 435–438 (1946).

Yule, G. Udny. Cumulative sampling: a speculation as to what happens in copying manuscripts. J. Roy. Statist. Soc. (N.S.) 109, 44–52 (1946).

Variations in old manuscripts are to a great extent due to copying errors and these are in turn frequently related to "danger spots" in the outward appearance. Since the error removes the danger spot, variations due to copying errors will in general be more stable than the original version. The author uses an admittedly greatly oversimplified model of a random game to study the probable development within so-called families of texts. The mathematics is elementary and the interest of the paper lies in conclusions which apparently differ greatly from commonly accepted views. It is stated that the criterion has been applied to a particular case with results contradicting the philologists' conclusions. *W. Feller* (Ithaca, N. Y.).

Fréchet, Maurice. Fondements des méthodes statistiques d'estimation. Portugaliae Math. 5, 137–141 (1946).

Remarks on the estimation of the value of an unknown parameter. The main emphasis is on the idea of a confidence interval. *J. L. Doob* (Urbana, Ill.).

Kruskal, William. Helmert's distribution. Amer. Math. Monthly 53, 435–438 (1946).

The joint distribution of sample mean and sample standard deviation for random samples of size n from a normal population is derived by induction on n. *H. Scheffé.*

Molina, Edward C. Some fundamental curves for the solution of sampling problems. Ann. Math. Statistics 17, 325–335 (1946).

The curves show the 10%, 25%, 50%, 75% and 90% of the incomplete beta function. The arguments used are C and N, where

$$\int_0^b x^C(1-x)^{N-C}dx=P\int_0^1 x^C(1-x)^{N-C}dx,$$

$P=0.10$, 0.25, 0.50, 0.75, 0.90. Curves are given for $C=0(1)10(2)20(5)50$, extending for $C\leq N\leq 100$. The curves are applied to binomial sampling using a priori probability. *J. W. Tukey* (Princeton, N. J.).

Finney, D. J. The frequency distribution of deviates from means and regression lines in samples from a multivariate normal population. Ann. Math. Statistics 17, 344–349 (1946).

Calculation of the variances, covariances, correlation coefficients and some partial correlation coefficients of deviations taken (i) from the means of multivariate samples, (ii) from regressions fitted by least squares to the data furnished by the sample. The results are stated for multivariate normal distributions but apply generally to any multivariate distribution with finite second moments. *J. W. Tukey* (Princeton, N. J.).

Stevens, W. L. Mathematical theory of some distributions used in statistics. Revista Fac. Ci. Univ. Coimbra 10, 247–288 (1942); 11, 85–102 (1943). (Portuguese)

Exposition of the distributions of chi-square, t and z, using much careful geometrical argument and supplementing this with detailed analytical developments. Application to linear regression and the analysis of variance of randomized blocks. *J. W. Tukey* (Princeton, N. J.).

Stevens, W. L. Statistical estimation. Theory of the estimation of two or more parameters, illustrated by the problem of the estimation of the frequencies of the genes of blood groups. Revista Fac. Ci. Univ. Coimbra 12, 23–104, 175–221 (1944). (Portuguese)

Stevens, W. L. Application of the χ^2 test to the analysis of variance. Revista Fac. Ci. Univ. Coimbra 13, 4–17 (1945). (Portuguese)

Exposition of the elementary analysis of variance, particularly the case where n_2 is large. *J. W. Tukey.*

Fernández Baños, O. Contribution to the study of Pearson's χ^2. Revista Mat. Hisp.-Amer. (4) 6, 66–83 (1946). (Spanish)

Expository article. *J. W. Tukey* (Princeton, N. J.).

Hsu, P. L. On the asymptotic distributions of certain statistics used in testing the independence between successive observations from a normal population. Ann. Math. Statistics 17, 350–354 (1946).

Let $(x_1, \cdots, x_N)$ be a random sample from a normal population with mean 0 and variance 1. Put

$$T=Q/S,\quad Q=\sum_{i=1}^{N}a_{ij}(x_i-\bar{x})(x_j-\bar{x}),\quad S=\sum_{i=1}^{N}(x_i-\bar{x})^2.$$

By a rotation Q and S can be simultaneously reduced to $Q=\sum_{r=1}^{N}\lambda_r y_r^2$, $S=\sum_{r=1}^{N}y_r^2$. Now assume that (a) $|\lambda_r|\leq 1$ for all r, (b) there exists a number $c>0$ independent of n such that $\sum_{r=1}^{N}(\lambda_r-\bar{\lambda})^2>c_n$, where $\bar{\lambda}=n^{-1}\sum_{r=1}^{N}\lambda_r$. Under these

conditions the author proves that Cramér's asymptotic expansion [Random Variables and Probability Distributions, Cambridge University Press, 1937] can be applied to the distribution function $G(x) = \Pr\{T \leqq \bar{\lambda} + z\}$. The author then considers three statistics Q_n of the above form which occur in recent works and shows that (a) and (b) are satisfied in each case. *H. Cramér* (New Haven, Conn.).

Wilks, S. S. Sample criteria for testing equality of means, equality of variances, and equality of covariances in a normal multivariate distribution. Ann. Math. Statistics 17, 257–281 (1946).

The likelihood ratio tests for the following three hypotheses about a k-variate normal population are calculated: (i) that the means are all equal, the variances are all equal and the covariances are all equal; (ii) that the variances are all equal and the covariances are all equal, with no assumptions about the means; (iii) that the means are all equal, given that the variances are all equal and the covariances are all equal. The distribution of the likelihood ratio statistic when the hypothesis is true is determined by some sophisticated moment calculations; in each case the distribution turns out to be that of a product of independent beta variables: a product of $k-1$ factors in cases (i) and (ii), one factor in case (iii). Wilks's theorem [same Ann. 9, 60–62 (1938)] relating the asymptotic distribution of the likelihood ratio statistics to the chi-square distribution is applied. Relations of (ii) and (iii) to other tests are pointed out. An illustration of the three tests is worked out on some data from college entrance examinations. *H. Scheffé.*

Tukey, John W., and Wilks, S. S. Approximation of the distribution of the product of beta variables by a single beta variable. Ann. Math. Statistics 17, 318–324 (1946).

The problem of finding a useful approximation to the distribution of a random variable distributed like a product of r' independent beta variables has arisen in connection with many likelihood ratio tests [for example, Wilks, Biometrika 24, 471–494 (1932); also the paper reviewed above]. The moments of such a random variable X may be written in the form

$$(*)\qquad E(X^g) = \prod_{i=1}^{r'} \{(\lambda+a_i)_g/(\lambda+b_i)_g\},$$

where $(a)_g = \prod_{j=0}^{g-1}(a+j)$ and $g = 0, 1, \cdots$. In this attack on the problem, the distribution of X is approximated by the distribution of V^r, where V is a beta variable with indices p and q. The three constants r, p, q are determined by fitting moments, but in a novel way: instead of fitting a few moments exactly, all moments (*) are approximated simultaneously, the approximation being such that, if the moments are expanded in powers of $1/\lambda$, the coefficients of the zeroth and first powers are correct and the second "nearly correct," for all moments. [It would be interesting to have some evidence of how good the approximation is on a probability scale (that is, if a 1 or 5% point is approximated by this method, what is the true "percent" of the approximation?). Correction: after the displayed expression second from the bottom of p. 318, and that first from the top of p. 319, insert the factor $\Gamma(p_i+q_i)/\Gamma(p_i)$.] *H. Scheffé.*

Haldane, J. B. S. The cumulants of the distribution of Fisher's 'u_{11}' and 'u_{31}' scores used in the detection and estimation of linkage in man. Ann. Eugenics 13, 122–134 (1946).

The author considers the distributions of certain scores useful in the analysis of genetic data. It appears that the distributions are sufficiently far from normal to require special methods in testing for significance. *C. P. Winsor* (Baltimore, Md.).

Mourier, Édith. Étude du choix entre deux lois de probabilité. C. R. Acad. Sci. Paris 223, 712–714 (1946).

The author investigates the problem of the determination of the one of the two probability densities f and g which gave rise to the sample $x_1, \cdots, x_n$. It is supposed that a priori probabilities exist that f or g was the actual density. If the test is based on the value assumed by the function $v = \sum \log\{f(x_i)/g(x_i)\}$, the criterion minimizing the probability of an incorrect choice is to choose f if $v > a$ and g if $v < a$, where a is evaluated in terms of m_1, m_2, σ_1, σ_2 defined below. With this criterion the probability of error is a function of n and of $\rho = (m_1 - m_2)/(\sigma_1 + \sigma_2)$, where

$$m_1 = \int \{\log f/g\} f dx, \quad \sigma_1^2 = \int \{\log f/g\}^2 f dx,$$

$$m_2 = \int \{\log f/g\} g dx, \quad \sigma_2^2 = \int \log\{f/g\}^2 g dx.$$

It is suggested that ρ, which determines the ease in distinguishing between f and g, is suitable as a definition of the statistical distance between probability densities. This distance is symmetric, positive if f and g are different, and is unaffected by a change of variable. The validity of the triangle inequality has not been investigated. If $f(x, \theta)$ is a probability density depending on the parameter θ, the statistical distance between $f(x, \theta)$ and $f(x, \theta_0)$ is (apart from terms of higher order in $(\theta - \theta_0)$) $\frac{1}{2}|\theta - \theta_0|$ multiplied by the square root of the "amount of information" as defined by R. A. Fisher. *J. L. Doob* (New York, N. Y.).

Radhakrishna Rao, C. Tests with discriminant functions in multivariate analysis. Sankhyā 7, 407–414 (1946).

The author first derives a general distribution theorem. Let x_{ir} $(r = 1, \cdots, s)$ be a set of independent observations on the variates x_i $(i = 1, \cdots, k)$, which are normally and independently distributed with unit variance, and which all have mean zero except for x_1. Let $b_{ij} = \sum_r (x_{ir} - \bar{x}_i)(x_{jr} - \bar{x}_j)$ and let b_i^{pq} be an element of the matrix inverse to the matrix of b_{pq}'s obtained by letting p, q vary from 1 to i. Finally, let $V_i = \sum_{p=1}^{i} \sum_{q=1}^{i} b_i^{pq} s \bar{x}_p \bar{x}_q$. Then the author obtains, for any r, the distribution of V_r and of $U = (1 + V_k)/(1 + V_r) - 1$, and shows that V_r and U are independently distributed. In particular, when $E(x_1) = 0$, $V_r(s-r)/r$ has the F-distribution with r and $s-r$ degrees of freedom, and $U(s-k)/(k-r)$ has the F-distribution with $k-r$ and $s-k$ degrees of freedom.

Statistics for testing several multivariate hypotheses are derived by the method of discriminant analysis, and their distributions under the null hypothesis found with the aid of the preceding lemma on distribution. The principal result is a test for the hypothesis that the means of p correlated variates are all equal to each other, when a sample of n observations on the p-dimensional population is available. The proposed statistic is V_{p-1}, defined as the root of a determinantal equation. Its distribution is shown to be that of V_r, with $r = p-1$ and $s = n$. [However, it may easily be inferred from equations (3.3) and (3.6), setting $m_{ii} = 1$, $m_{i,i+1} = -1$, $m_{ij} = 0$ for $j \neq i, i+1$, that the proposed test is identical with that obtained by taking the $p-1$ differences $x_i - x_{i+1}$ and then applying Hotelling's T^2-test to the hypothesis that the mean values of all the differences are zero.]

A test is proposed for the hypothesis that a given multivariate population is closer to one of two other multivariate

populations, but the hypothesis is never clearly defined. [In equation (2.7), the exponent of t should be $s-1$, not $s-1-(r-1)/2$. In equation (3.11), d^{ij} should be replaced by d_{ij}.] *K. J. Arrow* (New York, N. Y.).

Radhakrishna Rao, C., and Janardhan Poti, S. On locally most powerful tests when alternatives are one sided. Sankhyā 4, 439 (1946).

Jeffreys, Harold. An invariant form for the prior probability in estimation problems. Proc. Roy. Soc. London. Ser. A. 186, 453–461 (1946).

Let P and P' be cumulative distribution functions; then $\int((dP')^{\frac{1}{2}}-(dP)^{\frac{1}{2}})^2$ and $\int \log (dP'/dP)d(P-P')$ may be considered as measures of the difference between the distributions. The same differential geometry corresponds to each of these expressions, when an n-parameter family of distributions is considered, as to the expression arccos $\int(dP'dP)^{\frac{1}{2}}$ introduced by Bhattacharyya [Bull. Calcutta Math. Soc. 35, 99–109 (1943); these Rev. 6, 7]. The g_{ij} are given by the components of the information matrix, as was pointed out by Radhakrishna Rao [Bull. Calcutta Math. Soc. 37, 81–91 (1945); these Rev. 7, 464]. Thus $(|g_{ij}|)^{\frac{1}{2}}d\alpha_1 d\alpha_2 \cdots d\alpha_n$ is invariant and can be considered an invariant density of a priori probability. It proves to be unsatisfactory for scale parameters, however, and it is suggested that it be used only for the dimensionless parameters $\alpha_1, \alpha_2, \cdots, \alpha_n$ which may be assumed to accompany dimensional parameters of location λ and scale σ. This leads to an a priori density invariant under the transformations $\lambda'=\lambda+\sigma f(\alpha)$, $\sigma'=\sigma g(\alpha)$, $\alpha'=h(\alpha)$, where α stands for the n α_i's. *J. W. Tukey.*

Cramér, Harald. A contribution to the theory of statistical estimation. Skand. Aktuarietidskr. 29, 85–94 (1946).

Let $f(x_1, \cdots, x_n, \alpha_1, \cdots, \alpha_k)$ be the joint probability density of $x_1, \cdots, x_n$, which depends on the parameters $\alpha_1, \cdots, \alpha_k$, with $k<n$. Let $\alpha_1^*, \cdots, \alpha_k^*$ be unbiased estimates of $\alpha_1, \cdots, \alpha_k$, respectively, and $\|\lambda_{ij}\|$ their covariance matrix. Let $\|\lambda^{ij}\|$ be the inverse of $\|\lambda_{ij}\|$ and

$$\mu_{ij}=E((\partial(\log f)/\partial\alpha_i)(\partial(\log f)/\partial\alpha_j)).$$

It is proved that the "concentration ellipsoid"

$$\sum\lambda^{ij}(u_i-\alpha_i)(u_j-\alpha_j)=k+2$$

always contains within it the ellipsoid

$$\sum\mu_{ij}(u_i-\alpha_i)(u_j-\alpha_j)=k+2.$$

The concentration ellipsoid of $\alpha_1^*, \cdots, \alpha_k^*$ can coincide with this minimal ellipsoid when and only when $\alpha_1^*, \cdots, \alpha_k^*$ are (a) sufficient estimates of $\alpha_1, \cdots, \alpha_k$ and (b) are linear functions of $\partial(\log g)/\partial\alpha_i$, $i=1, \cdots, k$, where g is the density function of $\alpha_1^*, \cdots, \alpha_k^*$. *A. Wald* (New York, N. Y.).

Ruist, Erik. Standard errors of the tilling coefficients used in confluence analysis. Econometrica 14, 235–241 (1946).

Using Cramér's general formula for the sampling variance of a moment function [Mathematical Methods of Statistics, Princeton University Press, 1945, pp. 359 ff.; these Rev. 8, 39] the author has computed the large sample standard errors of the tilling coefficients used in constructing the bunch maps of Frisch's confluence analysis. With these results he illustrates with an example a suggestion of Frisch's that in bunch maps beams be replaced by standard error sectors. *C. C. Craig* (Ann Arbor, Mich.).

Girshick, M. A. Contributions to the theory of sequential analysis. II, III. Ann. Math. Statistics 17, 282–298 (1946).

[For part I see the same Ann. 17, 123–143 (1946); these Rev. 8, 44.] In part II a general solution of the sequential test problem [A. Wald, same Ann. 16, 117–186 (1945); these Rev. 7, 131] is given when the variate is discrete and takes on integral values over a finite range. Explicit expressions are given for the distribution of n, the sample size, and for the power function of the test. The method is applied to the sequential binomial probability ratio test.

In part III some results pertaining to the following sequential problem are given. A continuous variate x has a distribution known to belong to a one-parameter family $f(x, \theta)$. It is assumed that $f(x, \theta)$ admits a sufficient statistic for estimating θ, hence that $\log f(x, \theta)=u(x)v(\theta)+r(x)+w(\theta)$. It is further assumed that $v(\theta)$ is monotonic and that $f(x, \theta)$ satisfies certain regularity conditions. The author then shows how to obtain conjugate [see Wald, loc. cit.] pairs of distributions and shows that the distribution of n can be obtained in terms of these pairs of conjugate functions. It is also shown that the distribution of n, given one of a conjugate pair, is identical with the distribution of n, given the other member of the pair. *A. M. Mood.*

★Lotka, Alfred J. Population Analysis as a Chapter in the Mathematical Theory of Evolution. Essays on Growth and Form Presented to D'Arcy Wentworth Thompson, edited by W. E. Le Gros Clark and P. B. Medawar, pp. 355–385. Oxford, at the Clarendon Press, 1945. $6.00.

A review of the treatment of the so-called integral equation of renewal theory and an analysis of its implications for population theory. Several statistical illustrations are given. *W. Feller* (Ithaca, N. Y.).

De Lury, D. B. The analysis of Latin squares when some observations are missing. J. Amer. Statist. Assoc. 41, 370–389 (1946).

For experiments which consist of a number of 4×4 Latin squares, elegant methods of analysis are given when (1) one or more observations, (2) several columns, (3) one column, (4) two columns, (5) one column and one or more observations are missing. The analyses are based on the "missing plot" technique of Yates [Empire Journal of Experimental Agriculture 1, 129–142 (1933)]. *W. G. Cochran.*

Radhakrishna Rao, C. Confounded factorial designs in quasi-Latin squares. Sankhyā 7, 295–304 (1946).

Let k be the number of plots in a row or column of a quasi-Latin square, r the number of replications, kn the number of treatment combinations. The author discusses arrangements of the treatment combinations in quasi-Latin squares which satisfy the additional condition that the rows (columns) can be arranged in r sets of n rows (columns) such that each set contains a complete replication of the treatment combinations. The designs are of practical value only if at least the main effects remain unconfounded. Designs of this type have been constructed by R. C. Bose and K. Kishen [Sankhyā 5, 21–36 (1940); these Rev. 4, 222]. The author discusses existing solutions for such designs and adds new solutions. He finds necessary and sufficient conditions for the existence of a quasi-Latin square with r replications where certain specified effects are to be confounded in the rows (columns) of the replications of rows (columns). Some of his results can be extended to

asymmetrical designs. The last section of the paper is devoted to a discussion of quasi-Latin squares in varietal trials. The analysis of variance of the latter designs is given explicitly. *H. B. Mann* (Columbus, Ohio).

Radhakrishna Rao, C. On the most efficient designs in weighing. Sankhyā 7, 440 (1946).

With p objects to be weighed on a balance whose accuracy is virtually independent of load within appropriate limits, an increase in accuracy in each determination is possible without increasing the number n of weighing operations, if suitable combinations of objects are placed in the two pans. The design for such an experiment is represented by a matrix A whose elements are 0 or ± 1. For maximum efficiency $A'A$ must be a scalar, as shown by the reviewer [Ann. Math. Statistics **15**, 297–306 (1944); these Rev. **6**, 93]. The author shows that the maximum value of p for which fully efficient weighing experiments are possible is 2^s if $n=2^s(2m+1)$ or $n=2^s$, where m and s are integers, and full efficiency means that the variance of each determination is that of an individual weighing divided by n. Extensions of the problem are mentioned. *H. Hotelling.*

TOPOLOGY

Vigneron, L. Sur le problème des quatre couleurs: Théorie de la combinaison. C. R. Acad. Sci. Paris **223**, 705–707 (1946).

A special relation between three linear graphs is defined. It is shown that if two of the graphs are colorable in three colors, the third is. An application is made with one of the graphs an open or closed Kempe chain. *P. Franklin.*

Vigneron, Léopold. Remarques sur les réseaux cubiques de classe 3 associés au problème des 4 couleurs. C. R. Acad. Sci. Paris **223**, 770–772 (1946).

The author's method of combining colorations of maps is used to prove relations modulo 4 satisfied by the $+1$, -1 vertex indices associated with the coloration of a regular map in four colors. *P. Franklin* (Cambridge, Mass.).

Steinhaus, Hugo. Sur la division des ensembles de l'espace par les plans et des ensembles plans par les cercles. Fund. Math. **33**, 245–263 (1945).

The original "ham sandwich" theorems stated that (i) given three measurable sets in space there is a plane dividing each of their volumes in half, (ii) given three sets in the plane there is a circle containing half the area of each set. It is now proved that, if each of the three sets can be separated from the others by a plane, then the fraction $\frac{1}{2}$ may be replaced by any three chosen ratios. Conversely, no set of fractions except $(0, 0, 0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(1, 1, 1)$ can be obtained for all families of three sets. [During the suspension of publication of this volume, results generalizing the original theorem and covering special cases of the author's converse were obtained by A. H. Stone and the reviewer, Duke Math. J. **9**, 356–359 (1942); these Rev. **4**, 75.] *J. W. Tukey* (Princeton, N. J.).

Hadwiger, H. Mitteilung betreffend meine Note: Überdeckung einer Menge durch Mengen kleineren Durchmessers. Comment. Math. Helv. **19**, 72–73 (1946).

The author recently [same Comment. **18**, 73–75 (1945); these Rev. **7**, 215] communicated a proof of the surmise of K. Borsuk that a point set of diameter 1 in n-dimensional Euclidean space is coverable by means of $n+1$ sets each of diameter less than 1. The purpose of this note is to report that this proof is erroneous and to show that the theorem in question holds upon an added assumption, namely, that the solid of constant width (to which case the theorem as stated is reducible) has a regular boundary. *H. Blumberg* (Columbus, Ohio).

de Mira Fernandes, Aureliano. Funzioni continue sopra una superficie sferica. Portugaliae Math. **5**, 132–134 (1946).

By an elementary topological argument the author proves the following theorem. Let $f(P)$ be a continuous function defined on the surface of a sphere in Euclidean 3-space and let there be given a regular trihedron whose vertex O lies in the interior of the sphere. Then by a rotation about O the trihedron may be brought into such a position that the three values of f at the intersection points of the edges and the sphere are equal. *W. Fenchel* (Copenhagen).

Ostrowski, Alexandre. Nouvelle démonstration du théorème de Schoenflies pour les espaces à n dimensions. C. R. Acad. Sci. Paris **223**, 530–531 (1946).

A proof, based only on the Jordan separation theorem, of the theorem that the image in P_n of an open set in R_n, under a one-one continuous transformation Π, is itself an open set. That the inverse of Π is continuous was shown in an earlier note [same C. R. **223**, 229–230 (1946); these Rev. **8**, 49]. It is also shown that if a Jordan surface G and its interior region G_i lie in the domain of Π, then ΠG_i is the interior region of ΠG. *P. A. Smith* (New York, N. Y.).

Haupt, Otto. Vollständigkeitsprobleme bei geometrischen Ordnungen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1941**, 57–66 (1941).

The results of this paper are proved in a later paper which has already been reviewed [Ann. Mat. Pura Appl. (4) **23**, 123–148 (1944); these Rev. **7**, 468]. *J. H. Roberts* (Durham, N. C.).

Vicente Gonçalves, J. Sur une classe de frontières de domaines. Revista Fac. Ci. Univ. Coimbra **9**, 247–274 (1941).

The results of this paper are not new. The frontières de domaines considered fall in the class of boundary curves in a plane. (A boundary curve is a compact Peano continuum whose maximal cyclicly connected subsets are simple closed curves.) Under additional hypotheses it is shown that boundary points are arcwise accessible from the complement. *J. H. Roberts* (Durham, N. C.).

Knaster, Bronisław. Sur les coupures biconnexes des espaces euclidiens de dimension $n>1$ arbitraire. Rec. Math. [Mat. Sbornik] N.S. **19**(61), 9–18 (1946). (Russian. French summary)

There is constructed, for each $n>1$, an $(n-1)$-dimensional biconnected subset M of Euclidean n-space E_n such that (1) M cuts E_n and cuts E_n locally at each point of M (one recalls that a biconnected set cannot separate E_n); (2) M contains a point p such that the individual points of $M-p$ constitute the only quasi-components (a fortiori, the only connected subsets) of $M-p$. The construction follows. Let C denote the Cantor ternary subset of $0\leqq x\leqq 1$ on the x-axis of E_n, p the point of this axis at $x=3$. For each $c\epsilon C$, $S(c)$ denotes the surface of that $(n-1)$-dimensional

* The author's result, in a slightly more special form, was anticipated by Kakutani [Ann. of Math. (2) 43, 739–741 (1942); these Rev. 4, 111]

sphere whose diameter is the segment from c to p. Then $M=p+\sum p(c)$, where $p(c)$ is the point uniquely selected on $S(c)$, $c\epsilon C$, by the following procedure. First, the set C is resolved into a sum of mutually exclusive subsets each dense in C; thus $C=\sum C_a$, $0<a<1$, with $\bar{C}_a=C$ and $C_a\cdot C_{a'}$ vacuous when $a<a'$. Next one constructs the set $\{K_a\}$, $0<a<1$, of all those continua K_a which join a point interior to $S(1)$ to a point exterior to $S(0)$ but do not contain the point p. Now, to each $c\epsilon C$ there is a unique $a=a(c)$ such that $c\epsilon C_a$. Finally, $p(c)$ is defined as that one among all the points of the corresponding compact intersection $K_a\cdot S(c)$ whose coordinates are earliest in lexicographic order. It is shown, by adaptation of known results, that the set M may be constructed without appeal to the axiom of choice.

L. Zippin (Brooklyn, N. Y.).

Young, Gail S., Jr. Spaces congruent with bounded subsets of the line. Bull. Amer. Math. Soc. 52, 915–917 (1946).

A metric space S with distance xy and diameter $\delta\leqq\infty$ is congruent to a bounded subset of the real axis if and only if for any other metrization $d(x,y)$ of S with the same diameter δ points a, b exist for which $d(a,b)>ab$.

H. Busemann (Northampton, Mass.).

Valentine, F. A. Set properties determined by conditions on linear sections. Bull. Amer. Math. Soc. 52, 925–931 (1946).

Let S represent any set of points in n-dimensional Euclidean space ($n\geqq 2$) and L_{n-r} an $(n-r)$-dimensional hyperplane. Then the set $S\cdot L_{n-r}$ is defined as an $(n-r)$-dimensional linear section of S. The author proves that, if each $(n-1)$-dimensional linear section of S is closed and connected, then S is closed. But if each $(n-1)$-dimensional linear section is bounded and connected, then S is bounded and connected. This latter result is extended to a normed linear metric space. A characterization of a closed convex set S is given in terms of the two-dimensional linear sections of S.

V. W. Adkisson (Fayetteville, Ark.).

Sorgenfrey, R. H. Concerning continua irreducible about n points. Amer. J. Math. 68, 667–671 (1946).

The principal result is the following theorem. In order that the compact metric continuum M be irreducible about some n points, $n\geqq 2$, it is necessary and sufficient that, for every proper decomposition of M into $n+1$ continua, the sum of some n of these fails to be connected. (A finite number of subcontinua of M form a proper decomposition if their sum is M and no one of them is a subset of the sum of the others.)

J. H. Roberts (Durham, N. C.).

Arens, Richard. The space L^ω and convex topological rings. Bull. Amer. Math. Soc. 52, 931–935 (1946).

A commutative, convex, complete metrizable ring is exhibited which does not have an equivalent norm topology. The ring is the set L^ω of functions f belonging to L^p on $(0, 1)$ for all $p\geqq 1$. The neighborhoods of zero are the sets $U_{p,\epsilon}$ of $f\epsilon L^\omega$ such that $\|f\|_p<\epsilon$. It is shown that: (1) if U is a convex open set in L^ω containing 0 and $UU\subset U$, then $U=L^\omega$; (2) if R is a complete topological ring with a complete set $\{U\}$ of convex neighborhoods of 0 such that $UU\subset U$ and $P(z)=\sum_0^\infty a_nz^n$ represents an entire function, then $P(f)$, $f\epsilon R$, is a continuous mapping of R into itself. The nonexistence of an equivalent norm topology for L^ω follows immediately from (1), and also from (2) since $|\log x|\epsilon L^\omega$ while $\exp(|\log x|)\bar{\epsilon}L^\omega$.

L. W. Cohen.

Arens, Richard F. A topology for spaces of transformations. Ann. of Math. (2) 47, 480–495 (1946).

L'auteur étudie, sur l'ensemble C des applications continues d'un espace topologique A dans un espace topologique B, la topologie définie récemment par R. H. Fox [Bull. Amer. Math. Soc. 51, 429–432 (1945); ces Rev. 6, 278]; lorsque B est un espace uniforme, cette topologie n'est autre que celle de la convergence uniforme sur les sous-ensembles compacts de A; elle est bien connue dans les cas classiques. Citons les plus importants de ces résultats: lorsque C est munie de la topologie considérée, l'application $(f,x)\rightarrow f(x)$ de $C\times A$ dans B est continue si A est localement compact, et la topologie sur C est la moins fine ayant cette propriété [résultat publié pour la première fois par Fox, loc. cit.]; lorsque A est localement compact et B complet, C est complet; le théorème d'Ascoli sur les familles équicontinues se généralise sans difficulté. Le résultat le plus original de l'auteur est une condition nécessaire pour que C soit métrisable lorsque B est métrisable; le fait que cette condition est suffisante peut se démontrer beaucoup plus simplement que ne le fait l'auteur, en utilisant le critère général de A. Weil pour qu'une structure uniforme soit métrisable.

J. Dieudonné (São Paulo).

Myers, S. B. Equicontinuous sets of mappings. Ann. of Math. (2) 47, 496–502 (1946).

A subset T of the class F of continuous mappings of one space X into a metric space Y is equicontinuous if, at each x_0 in X, $f(x)\rightarrow f(x_0)$ uniformly for all f in T, as $x\rightarrow x_0$. Let F have that topology in which convergence in F means uniform convergence on every compact subset of X, the "co. o." or k-topology. Conditions are sought insuring that such equicontinuous T have a compact closure in F, for example: X connected, Y locally compact, complete metric, and $\overline{T(x)}$ compact in Y for some x in X. Equicontinuous sets of homeomorphisms (for example, isometries) are considered, and in all cases converses showing minimal properties of the k-topology are proved. The methods are somewhat like those of van Dantzig and van der Waerden [Abh. Math. Sem. Hamburgischen Univ. 6, 367–376 (1928)] who prove the group of isometries of a connected locally compact metric space to be a locally compact group; but the present results are much more general. [Parts of this paper overlap with the reviewer's thesis [Harvard, 1945] in which k-topologized equicontinuous groups of homeomorphisms and families of functions, with values in uniform spaces, are included; cf. also the preceding review.]

R. Arens.

Hewitt, Edwin. On two problems of Urysohn. Ann. of Math. (2) 47, 503–509 (1946).

L'auteur complète les résultats d'Urysohn [Math. Ann. 94, 262–295 (1925)] sur les espaces topologiques où toute fonction réelle continue est nécessairement constante. En appliquant le procédé de condensation d'Urysohn [loc. cit.] à un espace construit de manière analogue à l'espace régulier et non complètement régulier de Tychonoff [Math. Ann. 102, 544–561 (1929)] il montre qu'il existe des espaces réguliers, de puissance quelconque, sur lesquels toute fonction réelle continue est constante; le même procédé de condensation, appliqué à un espace dénombrable convenablement construit, lui permet d'obtenir un espace dénombrable où toute fonction réelle continue est constante, et qui est complètement séparé (c'est-à-dire que deux points quelconques peuvent être séparés par des voisinages fermés de ces

points). Relativement aux axiomes de séparation actuellement connus, ces résultats ne peuvent être améliorés.
J. Dieudonné (São Paulo).

Kalisch, G. K. On uniform spaces and topological algebra. Bull. Amer. Math. Soc. 52, 936–939 (1946).

The author considers an approach to uniform spaces which develops an analogy to metric spaces ("generalized metric function") which differs somewhat from that chosen by A. Weil. The equivalence of the two formulations is shown. In applications to groups with operators from a topological field, the "generalized metric" can be made to resemble a norm except for the lack of a homogeneity condition. *R. Arens* (Princeton, N. J.).

Leray, Jean. Propriétés de l'anneau d'homologie de la projection d'un espace fibré sur sa base. C. R. Acad. Sci. Paris 223, 395–397 (1946).

Leray, Jean. Sur l'anneau d'homologie de l'espace homogène, quotient d'un groupe clos par un sous-groupe abélien, connexe, maximum. C. R. Acad. Sci. Paris 223, 412–415 (1946).

The author continues his study of the cohomology ring R of a map $f:X\rightarrow Y$ over a ring A [same C. R. 222, 1366–1368, 1419–1422 (1946); these Rev. 8, 49]. Assuming that R has finite rank over A, three kinds of Poincaré polynomials are defined. If X and Y are orientable manifolds and f is a fibre map, then duality relations in R analogous to the Poincaré duality theorem are stated. If, furthermore, A is a field and the fibres are connected then R is the direct sum of the cohomology ring of the base space Y and the cohomology ring of the fibre. In this case relations between the various Poincaré polynomials are stated.

In the second paper the method is applied to the case when X is a closed Lie group and Y is a homogeneous space obtained by factoring X by a subgroup. The cohomology ring of Y is then written down in a number of cases when X is a classical group. No proofs are indicated.
S. Eilenberg (Bloomington, Ind.).

Cairns, Stewart S. The triangulation problem and its role in analysis. Bull. Amer. Math. Soc. 52, 545–571 (1946).

Survey of the triangulation problem in its most general formulation (proposed by the author and established by him for differentiable spaces). Is it possible to subdivide every locally polyhedral space into the cells of a complex? New results relating to simultaneous triangulations of two manifolds which lie one over the other. The regularity problem (that is, the question of whether every topological manifold can be smoothed into a differentiable manifold) is equivalent to the triangulation problem if the dimension is less than 5. Applications to the analysis of manifolds.
H. Freudenthal (Amsterdam).

Dedecker, Paul. Sur la notion d'involution et la formule de Zeuthen. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 30 (1944), 58–66 (1945).

The author extends his previous work [same Bull. (5) 29, 680–687 (1943); these Rev. 7, 216] to the case of "covering surfaces with folds." *M. H. Heins* (Providence, R. I.).

Hirsch, Guy. Sur la signification topologique des axiomes de la géométrie projective. C. R. Acad. Sci. Paris 223, 528–530 (1946).

This paper studies the following situation. A $2n$-dimensional manifold V^{2n} is given and in it a family (each two members of which are homeomorphic) of n-dimensional manifolds M^n which form a projective geometry in the sense that any two of the manifolds M^n intersect in a point and through any two points of V^{2n} there passes exactly one M^n. The author asks what can be said about V^{2n} and M^n and asserts that M^n is homeomorphic to an n-sphere. He describes three other situations which are equivalent to the one mentioned above and remarks that the Cayley numbers give an example of a V^{16} containing a family of manifolds M^8 satisfying his conditions. *D. Montgomery.*

Freudenthal, Hans. Verbesserungen und Berichtigungen. Ann. of Math. (2) 47, 829–830 (1946).

The corrections are to the author's papers in the same Ann. (2) 42, 1051–1074 (1941); 43, 261–279, 580–582 (1942); these Rev. 3, 198, 315; 4, 88.

Freudenthal, Hans. Der Einfluss der Fundamentalgruppe auf die Bettischen Gruppen. Ann. of Math. (2) 47, 274–316 (1946). [MF 16334]

H. Hopf studied the influence of the fundamental group of a polytope on the second homology group [Comment. Math. Helv. 14, 257–309 (1942); these Rev. 3, 316]. His results have led to several independent discoveries of a more general theorem. Given any group π and any Abelian coefficient group, one can construct algebraically the nth "cohomology" groups $H^n(\pi, G)$. If X is any arcwise connected topological space for which the homotopy groups $\pi_2=\cdots=\pi_{r-1}=0$ vanish, the nth (singular) cohomology group of the space, with coefficients in G, is determined algebraically by the fundamental group $\pi=\pi_1(X)$ as $H^n(\pi, G)$, provided $n<r$. If $n=r$, a factor group of $H^r(X, G)$, the group of annihilators of spherical cycles, is also determined as $H^r(\pi, G)$. By duality, the corresponding homology groups of the space are also determined. This is the Eilenberg-MacLane form of the theorem [Proc. Nat. Acad. Sci. U. S. A. 29, 155–158 (1943); Ann. of Math. (2) 46, 480–509 (1945); these Rev. 4, 224; 7, 137]. Hopf gave a direct determination of the homology groups in the more general case when π is regarded as a group of simplicial transformations (without fixed points) of a complex, which could in particular be the covering complex of the given space X [Comment. Math. Helv. 17, 39–79 (1945); these Rev. 6, 279]. The paper under review proves the theorem above, for X a polytope, by direct construction of the homology groups from π. The construction uses a representation of π by a sequence of free groups. It is proved that the result of the construction is independent of the choice of this sequence. The cohomology groups and the cup products in the polytope are also determined by explicit duality, using extensions of some of the author's previous results on duality in groups. The author's construction of homology groups is close to that used by Hopf in his second paper.
S. MacLane (Cambridge, Mass.).

Eckmann, Beno. Der Cohomologie-Ring einer beliebigen Gruppe. Comment. Math. Helv. 18, 232–282 (1946).

This paper studies the effect of the fundamental group of a polytope upon higher cohomology groups [see the preceding review]. For any abstract group one constructs in invariant fashion an associated complex and the related cohomology group. The definitions are directly equivalent to those used by Eilenberg and MacLane [reference in the preceding review]. The proof that these constructions give the cohomology group of the complex is new, in that it

operates not with the original complex but with the covering complex. In this connection the author defines an abstract covering of an abstract complex. He demonstrates that the cohomology ring of the complex is determined algebraically by π, up to dimension r. This ring is computed explicitly in the case when π is a free Abelian group with a finite set of generators. *S. MacLane.*

Whitehead, J. H. C. Note on a previous paper entitled "On adding relations to homotopy groups." Ann. of Math. (2) **47**, 806–810 (1946).

The paper cited [same Ann. (2) **42**, 409–428 (1941); these Rev. **2**, 323] is concerned with the relations between $\pi_n(X)$ and $\pi_n(X^*)$, where X^* is obtained from X by "adjoining" a certain number of n-cells. In the case $n=2$, which is complicated by the noncommutativity of the fundamental group, the author introduced a group h_{π_1} whose elements are constructed formally from the elements of $\pi_1(X)$ and symbols representing the 2-cells of X^*-X. In the present note it is pointed out that h_{π_1} is actually isomorphic to the relative homotopy group $\pi_2(X^*, X)$ and that the theory of the case $n=2$ can be simplified accordingly. If X is the 1-dimensional skeleton of X^*, this group ties in with the Reidemeister Überlagerung. *R. H. Fox.*

Whitehead, George W. On families of continuous vector fields over spheres. Ann. of Math. (2) **47**, 779–785 (1946).

It is well-known that over spheres S^{2n} of dimension $2n$ there exist no continuous fields of nonzero tangent vectors (in short: vector fields). B. Eckmann [Comment. Math. Helv. **15**, 1–26 (1943); these Rev. **4**, 173] and the author [same Ann. (2) **43**, 132–146 (1942); these Rev. **3**, 197] proved that over spheres S^{4n+1} there exist no two everywhere independent vector fields. The author now proves that over spheres S^{8n+3} there exist no four everywhere independent vector fields. The proof is based on known results on homotopy groups of spheres and facts from the theory of fiber-spaces and involves the construction of several specific elements of the (relative) homotopy groups of coset spaces of the orthogonal group. As an application it is shown that an S^n $(n>k)$ cannot be a k-sphere bundle over any polyhedron if $k=2m$, $4m+1$ or $8m+3$.

H. Samelson (Ann Arbor, Mich.).

GEOMETRY

Fejes, L. Über eine Abschätzung des kürzesten Abstandes zweier Punkte eines auf einer Kugelfläche liegenden Punktsystems. Jber. Deutsch. Math. Verein. **53**, 66–68 (1943).

The author proves the following theorem. Let there be given n points on the surface of the unit sphere. Then there always exist two of them those distance is not greater than $\{4-\csc^2 \frac{1}{6}\pi n/(n-2)\}^{\frac{1}{2}}$. For $n=3$, 4, 6 and 12 the limit is exact. *P. Erdös* (Syracuse, N. Y.).

Constantinescu, G. G. On some geometrical theorems. Pozitiva **1**, 257–263 (1941). (Romanian. French summary)

[Cf. D. Pompeiu, Bull. Math. Phys. Éc. Polytech. Bucarest **11**, 29–34 (1940); these Rev. **7**, 68.] Le théorème pour le carré donné par M. Pompeiu reste vrai pour un parallélogramme et, en général, M étant un point quelconque dans le plan d'un contour polygonal fermé $M_1M_2\cdots M_n$, avec les n longueurs MM_i on peut toujours former une ligne polygonale fermé si au moins une des deux conditions $M_{i-1}M_i=M_iM_{i+1}$, $M_{i-1}M_{i+1}=M_iM_{i+2}$ est satisfaite.

Author's summary.

Abellanas, Pedro. Analytic structure of the open segment defined by Hilbert's postulates of incidence and order. Revista Mat. Hisp.-Amer. (4) **6**, 101–126 (1946). (Spanish)

It is well known that, on the basis of Hilbert's postulates of incidence, order and parallelism, addition and multiplication of points on a line can be defined so that these elements form a noncommutative field. In this paper the two operations are defined for points of an open segment when only the postulates of incidence and order are assumed and the algebraic structure of the resulting system is studied. After proving the necessary theorems of Desargues type, the author considers an open segment seg. $(-I, I)$, a fixed interior point 0, and a point 1 not on the line of the segment. Let P, Q be the respective intersections of seg. $(-I, 1)$ and seg. $(1, I)$ with the lines I, 2 and $-I$, 2, where 2 is a selected interior point of seg. (0, 1). For A, $B\epsilon$seg $(-I, I)$ the sum $A+B$ is defined as follows. If $B\epsilon$seg. $(0, I)$ join A with 1 and intersect with seg. (I, P) in point 3, while if $B\epsilon$seg. $(-I, 0)$, 3 is the intersection of seg. $(1, A)$ with seg. $(-I, Q)$. Joining B with 2, the line B, 2 intersects seg. $(-I, 1)+$seg. $(1, I)$ in 4. If line 3, 4 cuts seg. $(-I, I)$, the intersection is $A+B$. It is proved that $A+B$ exists and is independent of the choice of points 1 and 2. The equations $A+X=B$, $X+A=B$ are shown to have unique solutions, and so each element has right and left inverses (with respect to addition) which are proved equal. Representing points of seg. $(0, I)$ generically by A, then those of seg. $(-I, 0)$ are represented by $-A$ and it is proved that addition is commutative for two points of different signs and not commutative for points of the same sign. Associativity relations are obtained, e.g., $(A+B)+C=A+(B+C)$, but $(A+B)-C\neq A+(B-C)$. The reader is referred to the paper for the definition of multiplication, which is proved associative but not commutative. *L. M. Blumenthal.*

Kopeikina, L. Decompositions of projective planes. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 495–526 (1945). (Russian. English summary)

A study of free planes was begun by the reviewer [Trans. Amer. Math. Soc. **54**, 229–277 (1943); these Rev. **5**, 72]. This paper extends those investigations. The free product of partial planes is defined as the free extension of the set-theoretical sum of the partial planes. The rank of a finite partial plane is $2(P+L)-I$, where P, L and I are, respectively, the number of points, lines and incidences. The rank is invariant under free extension and contraction. Thus the adjunction of a point not on any line, or a line not through any point, increases the rank by two, and either of these is equivalent to adjoining two points on a line already in the finite partial plane. Hence independent points and independent lines are not the "atoms" which generate free planes, but are "diatomic molecules" containing two free factors of rank one. The terms "free product," "free factor" and the name "rank" for the quantity $2(P+L)-I$ are additions to the reviewer's terminology.

There are three theorems of some importance. The first shows that a subplane of a free plane is a free plane, dispensing with the requirement of the reviewer's theorem that the number of generators is finite. The second is that every subplane of a free product of planes is the free product of subplanes of the factors and a free plane. A third gives the essential uniqueness of free decompositions into nondecomposable components. Two decompositions into nondecomposable components will be such that appropriate free extensions of the nonfree components will be pairwise identical, and the free components will have the same rank. However, an example shows that not every plane possesses a decomposition into nondecomposable components. A final example shows that a locally free plane (a plane whose finitely generated subplanes are free) need not itself be free.

M. Hall, Jr. (Columbus, Ohio).

Hager, Anton. Symmetrische Inzidenztafeln finiter Geometrien. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1943, 25–47 (1944).

Let the points of a finite plane be numbered $P_1, \cdots, P_N$, where $N=n^2+n+1$, and the lines numbered $L_1, \cdots, L_N$. A table with rows numbered 1 to N from top to bottom, and columns numbered 1 to N from left to right, is the incidence table for a plane if a mark is put in the cell in the rth row and sth column if and only if the point P_r lies on the line L_s. A similar table can be constructed for a 3-space giving incidences between points and planes. Numbering points and lines (or planes) appropriately the table will be symmetric with respect to the upper left-lower right diagonal if and only if the plane (3-space) is self-dual. The author gives a method, using a polarity, of constructing such a symmetric table and illustrates it with an example. If a plane on proper renumbering possesses a collineation which permutes $P_1, \cdots, P_N$ and $L_1, \cdots, L_N$ cyclically, the table is symmetric with respect to the upper right-lower left diagonal. The author shows how this may be done for Desarguesian planes, and also for 3-spaces, and illustrates with examples.

M. Hall, Jr. (Columbus, Ohio).

★Malengreau, Julien. Étude Critique du Théorème de Pythagore. 2d ed. F. Rouge & Cie., Lausanne, 1945. 127 pp.

This booklet is composed of six notes, of which the first two are devoted to a study of the Pythagorean theorem in the geometry of the rational plane which is constructed (by the aid of six postulates) from the vertices of an equilateral triangle. The author refers to the Pythagorean theorem, interpreted as expressing a relation between lengths, as the first theorem, while the interpretation in terms of surfaces is called the second theorem. To show that these are essentially different the first is derived from postulates which do not suffice to prove the second. The theorems are expressed in terms of hexangle triangles (i.e., triangles in which one angle is 60°) rather than right triangles. The third note is concerned with a development of geometry in which the Stewart four-point relation plays a fundamental role, while the fourth note uses this relation to obtain the special Euclidean $(n+2)$-point relation satisfied by the distances of a point P of the space from the vertices of an equilateral $(n+1)$-tuple. No reference is made to the literature for general $(n+2)$-point relations. The work is in the spirit of the author's Essai sur les Fondements de la Géométrie Euclidienne, Lausanne, 1938.

L. M. Blumenthal (Columbia, Mo.).

Chiang, L. F. A matrix theory of circles and spheres. Acad. Sinica Science Record 1, 257–262 (1945).

With an oriented circle in E_2 with center x, y and radius r associate the matrix

$$X=\begin{pmatrix} r+x & y \\ y & r-x \end{pmatrix}.$$

The facts that the determinant of X is $r^2-x^2-y^2$ and that the trace of X is $2r$ permit one to derive many properties of circles and Laguerre geometry in a simple way. Similarly, the geometry of spheres in E_3 may be based on the study of the Hermitian matrices

$$\begin{pmatrix} r+x & y+iz \\ y-iz & r-x \end{pmatrix},$$

where r is the oriented radius and (x, y, z) is the center of a sphere.

H. Busemann (Northampton, Mass.)

Lauffer, R. Beweis des Morleyschen Dreieckssatzes. Deutsche Math. 7, 405 (1944).

Lauffer, R. Eine Dualisierung der Brocard'schen Punkte des ebenen Dreiecks. Deutsche Math. 7, 406–414 (1944).

Voderberg, H. Über ein- und umbeschriebene Parallelogramme der Ellipse. Deutsche Math. 7, 172–177 (1943).

Seifert, L. Einige Bemerkungen über die kubische Involution auf einer Ellipse. Časopis Pěst. Mat. Fys. 70, 104–118 (1941). (Czech. German summary)

Three points on the ellipse

$$x=a(1-t^2)/(1+t^2), \quad y=2bt/(1+t^2)$$

are said to belong to a cubic involution if their parameters are the roots of a cubic equation such as $k^2t^3-t+\lambda(t^2-k^2)=0$, where λ varies while k is a constant. In this case the sides of the triangle touch a fixed ellipse, the circumcenter traces another ellipse and the circumcircle envelops a bicircular quartic. In particular, if a triangle is inscribed in the given ellipse and circumscribed about the concentric circle of radius $ab/(a+b)$, its circumcircle touches both the concentric circles of radii a and b. Again, if a triangle is inscribed in the ellipse and circumscribed about the circle with center at the foot of a directrix and radius $ab^2/(a^2-b^2)$, then its circumcircle touches the auxiliary circle (of radius a) and passes through the corresponding focus.

H. S. M. Coxeter (Notre Dame, Ind.).

Dolaptschijew, B. Über projektive Kegelschnittsysteme. Acta Univ. Szeged. Sect. Sci. Math. 11, 17–18 (1946).

Pour que les coniques de deux faisceaux linéaires ponctuels puissent être associées de façon que leurs points communs (supposés en général distincts) décrivent une conique, il faut et il suffit que ces faisceaux aient une conique commune. La correspondance est une homologie des paramètres. La démonstration est facile. Le cas où les points variables ne sont jamais tous distincts serait plus difficile mais n'est pas examiné ici.

L. Gauthier (Nancy).

Cattaneo, Paolo. Su una particolare famiglia di parabole. Atti Mem. Accad. Sci. Padova. Mem. Cl. Sci. Fis.-Mat. (N.S.) 57, 177–185 (1941).

Halteman, A. E. A geometric approach to the covariants of a cubic. Amer. Math. Monthly 53, 517–520 (1946).

Convex Domains, Integral Geometry

Olovjanischnikoff, S. Ueber eine kennzeichnende Eigenschaft des Ellipsoides. Leningrad State Univ. Annals [Uchenye Zapiski] 83 [Math. Ser. 12], 114–128 (1941). (Russian. German summary) [MF 16492]

Let a plane section dividing a convex body into volumes whose ratio is $\lambda:1$ be called a λ-section. The author's main theorem is that if, for some $\lambda>1$, all the λ-sections of a convex body have central symmetry then it is an ellipsoid. This improves a result due to H. Brunn [cf. W. Blaschke, Vorlesungen über Differentialgeometrie, vol. 2, Springer, Berlin, 1923, §§ 44, 84], who assumes symmetry of all plane sections and smoothness of the surface. The proof rests on a lemma: the family of all λ-sections of a convex body has characteristics at the centroids of the sections. The theorem is generalized to n dimensions by induction.

The following are some of the applications made. (I) If an absolutely continuous probability-distribution in n dimensions determines on each straight line a symmetrical and strictly unimodal distribution, then its surfaces of constant density are concentric homothetic ellipsoids. (II) For a piece F of a convex surface, if all planes near enough to some support plane T of F cut F, if at all, in centrally symmetrical curves, then F coincides near T with part of a quadric. [Under certain differentiability assumptions this was proved by Blaschke, loc. cit.] *H. P. Mulholland.*

Olovianishnikoff, S. Généralisation du théorème de Cauchy sur les polyèdres convexes. Rec. Math. [Mat. Sbornik] N.S. 18(60), 441–446 (1946). (Russian. French summary)

If (in E^3) the closed convex surface F is isometric to the convex polyhedron P (in the sense that geodesic distances of corresponding pairs of points on F and P are equal), then F is congruent to P. The main step is a proof without differentiability hypotheses of: if the spherical image of the open subset W of a convex surface has positive measure, then W is not isometric to a plane set. *H. Busemann.*

Olovianishnikoff, S. On the bending of infinite convex surfaces. Rec. Math. [Mat. Sbornik] N.S. 18(60), 429–440 (1946). (Russian. English summary)

Let F be the total boundary of an unbounded convex solid in E^3 and denote by F_λ the surface homothetic to F in the ratio $l:1$ from a fixed point O. For $l\rightarrow\infty$ the surface F_λ tends to a convex cone $K(F)$ whose total curvature equals that of F. A ray R on F is a set which, with the geodesic distances on F, is isometric to a Euclidean ray. As x traverses R toward ∞ the direction of the "right" semitangent of R at x converges to the direction of a generator $L(R)$ of $K(F)$, which is the limit of the rays which correspond to R on the F_λ. The following theorems are proved. (1) Let F be isometric to an (unbounded) convex polyhedron P with a finite number of vertices and edges and the same orientation as F. If corresponding rays R_F and R_P on F and P and a translation τ of E^3 exist such that τ carries $K(F)$ into $K(P)$ and $L(R_F)$ into $L(R_P)$ then F goes into P by a translation of E^3. (2) Let R be a ray in F, K^* a cone with the same total curvature as F and R' a given generator of K^*. There exists a convex surface F^* isometric to F with the same orientation as F, $K^*=K(F^*)$ and $R'=L(R^*)$, where R^* is the image of R on F^*.
H. Busemann (Northampton, Mass.).

Alexandroff, A. D. On the metric of a convex surface in a space of constant curvature. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 411–413 (1946).

Define the angle α between two segments S_1, S_2 (that is, isometric images of Euclidean segments) in a metric space with distance xy at a common end point p as follows. Let $x_i\neq p$ and denote by φ the angle opposite x_1x_2 in a Euclidean triangle with sides px_i and x_1x_2. Then $\alpha=\lim\inf\varphi$ as $x_i\rightarrow p$ in such a way that $x_iS_i/x_ip_i\rightarrow 0$. Let a convex surface S mean either the complete boundary B of a convex set or an open subset of B any two points of which can be connected by a shortest line in S. A metric space M is isometric to a convex surface in a simply connected space of constant curvature K if and only if (1) M is homeomorphic to an open region on a sphere; (2) any two points of M can be connected by a segment; (3) every point p of M has a neighborhood U such that, for any triangle in U (that is, a set of points a_1, a_2, a_3 and segments connecting them), the relation $\alpha_1+\alpha_2+\alpha_3-\pi>K\tau$ holds, where α_i is the angle at a_i and τ is the area of the Euclidean triangle with sides a_ia_k.
H. Busemann (Northampton, Mass.).

Pogorelov, A. A theorem regarding geodesics on closed convex surfaces. Rec. Math. [Mat. Sbornik] N.S. 18(60), 181–183 (1946). (Russian. English summary) [MF 16684]

It is known that on a surface whose Gaussian curvature is everywhere not less than k_0 the geodesic distance between any two points is not greater than $\pi/\sqrt{k_0}$. In this paper a complementary theorem is proved: if S is a closed convex surface whose Gaussian curvature is nowhere greater than k_0 and l is a geodesic on S of length less than $\pi/\sqrt{k_0}$, then l is the shortest curve on S joining its endpoints.
H. Wallman (Cambridge, Mass.).

Knothe, Herbert. Über Beziehungen zwischen der Liniengeometrie und der Theorie der konvexen Körper. Volumen und mittlere Breite im R_n. Deutsche Math. 7, 146–151 (1943).

For every twice continuously differentiable closed convex surface K in E_3 there are congruences of lines whose spherical images cover the unit sphere exactly once in such a way that the tangent planes of K intersect the lines of the congruence normal to them in points p which have constant distance from the center of their respective lines (Strahlmittelpunkte). Among these congruences there is exactly one, N, whose lines are the normals of a convex surface. In the case of N the points p form a closed surface F which contains K in its interior. Call V the volume of a convex body of maximal volume that is contained in F and contains K. Then $V\leqq M^3/48\pi^2$, where M is the integral over K of the mean curvature of K. The equality sign holds only for the sphere. The results are extended to E_n.
H. Busemann (Northampton, Mass.).

Fejes, László. Eine Bemerkung über die Bedeckung der Ebene durch Eibereiche mit Mittelpunkt. Acta Univ. Szeged. Sect. Sci. Math. 11, 93–95 (1946).

The author proves the following theorem. Let B be a convex region with a center, which is not an ellipse. Then there exists a covering of the plane by regions congruent to B whose centers form a lattice and whose density is less than $2\sqrt{3}\pi/9$. In case B is an ellipse the density is at least $2\sqrt{3}\pi/9$. Several similar problems are investigated.
P. Erdös (Syracuse, N. Y.).

Hadwiger, H. Separierbarkeit ebener Eibereiche durch eine Gerade. Experientia 2, 362 (1946).

The author proves the following theorem. If the sum of the circumferences of a finite number of convex domains is smaller than the circumference of the convex envelope then there is a straight line which separates the convex domains.

P. Erdös (Syracuse, N. Y.).

Inzinger, R. Über die Scheiteltangenten von Eilinien. Österreich. Ing.-Arch. 1, 135 (1946).

If the closed convex curve C is not a circle then the tangents of C at those points where the radius of curvature is stationary are never all tangent to the same circle.

H. Busemann (Northampton, Mass.).

Santaló, L. A. Convex regions on the n-dimensional spherical surface. Ann. of Math. (2) 47, 448–459 (1946).

Généralisation de résultats dus à H. W. E. Jung [J. Reine Angew. Math. 123, 241–257 (1901)], W. Blaschke [Jber. Deutsch. Math. Verein. 23, 369–374 (1914)], P. Steinhagen [Abh. Math. Sem. Hamburgischen Univ. 1, 15–26 (1922)], relatifs à la sphère minimum circonscrite à un ensemble de diamètre D, et à la sphère maximum inscrite à un ensemble d'épaisseur B dans l'espace euclidien à n dimensions. L'auteur établit pour des ensembles situés sur une sphère à n dimensions, plongée elle-même dans un espace euclidien à $n+1$ dimensions, des formules analogues à celles des auteurs ci-dessus et les contenant par un passage à la limite de la n-sphère au R_n. Il utilise ces résultats pour étendre au cas de n quelconque le théorème suivant établi pour $n=2$ par R. M. Robinson [Bull. Amer. Math. Soc. 44, 115–116 (1938)]. Soit K une région convexe de la sphère unitaire n-dimensionnelle $S_{n,1}$. Il y a toujours une surface sphérique à $n-1$ dimensions sur $S_{n,1}$, de rayon r donné par $\tan r=((n+1)/2n)^{-\frac{1}{2}}$, qui, ou bien est contenue dans K, ou bien n'a aucun point commun avec K ni avec la région symétrique de K par rapport au centre de $S_{n,1}$. Cette valeur de r est exacte.

Voici les inégalités obtenues par l'auteur pour K de diamètre sphérique D sur $S_{n,1}$ et sa sphère circonscrite de rayon sphérique R. Si $\cos R\geqq(n+1)^{-\frac{1}{2}}$ on a $\cos 2R\leqq\cos D\leqq\{(n+1)\cos^2 R-1\}/n$. Si $0\leqq\cos R\leqq(n+1)^{-\frac{1}{2}}$, on a

$$\cos 2R\leqq\cos D\leqq\frac{(n+1)\cos^2 R-1}{1+(n-1)\cos^2 R}$$

pour n impair,

$$\cos 2R\leqq\cos D \leqq\frac{(n+1)\cos^2 R-1}{\{1+(n+1)\cos^2 R\}^{\frac{1}{2}}\{1+(n+1)(n-2)(n+2)^{-1}\cos^2 R\}^{\frac{1}{2}}}$$

pour n pair. Des formules duelles analogues relient l'épaisseur B d'un ensemble K au rayon sphérique r de sa sphère inscrite.

E. Blanc (Clermont-Ferrand).

Santaló, L. A. On the probable distribution of corpuscles in a body, deduced from the distribution of its sections, and analogous problems. Revista Unión Mat. Argentina 9, 145–164 (1943). (Spanish) [MF 16815]

In a plane convex domain K there are N congruent convex figures distributed at random (in the familiar sense of integral geometry). It is shown that the average number of figures intersected by a chord of K is Nu/U, where u and U are, respectively, the perimeters of the figures and of K. The mean density of the intersection of a chord with the N figures on that chord is uD/π, where D is the density of the N figures in K. It is shown that this result can be considered as a generalization of a result of Pólya on the mean visible distance in a forest [Arch. Math. Phys. (3) 27, 135–142 (1918)]. Several other related quantities are computed and the results generalized to three dimensions.

W. Feller (Ithaca, N. Y.).

Neumann, B. H. On an invariant of plane regions and mass distributions. J. London Math. Soc. 20, 226–237 (1945).

L'auteur donne les définitions suivantes: (1) le demi-plan non-négatif $H(P, \varphi)$ attaché à un point P et à une demi-droite d'angle polaire φ issue de ce point est l'ensemble balayé par une demi-droite issue de P dont l'angle polaire varie de φ à $\varphi+\pi$ (bornes comprises). (2) Étant donnée dans le plan une distribution de masses telle que la masse $m(P, \varphi)$ contenue dans le demi-plan non-négatif $H(P, \varphi)$ existe et soit fonction continue de P et φ, on appelle $\rho(P, \varphi)$ le rapport $m(P, \varphi)/M$ (M masse totale dans tout le plan). Le point P restant fixe, $\rho(P, \varphi)$ admet un minimum lorsque φ varie de 0 à 2π. Ce minimum passe par un maximum μ lorsque P décrit tout le plan. Un point P_0 où μ est atteint est dit point extrêmal; les directions issues de P_0 pour lesquelles μ est atteint sont qualifiées d'extrêmales ainsi que les demi-plans correspondants.

L'auteur montre que pour une distribution quelconque, il existe un point extrêmal au moins dont les demi-plans extrêmaux couvrent tout le plan (le recouvrement peut alors être obtenu par trois de ces demi-plans). Il en déduit que pour une distribution quelconque (qui peut être en particulier une distribution superficielle ou linéaire homogène), on a $\frac{1}{3}\leqq\mu\leqq\frac{1}{2}$, la borne $\frac{1}{2}$ étant atteinte pour une distribution à centre de symétrie, et la borne $\frac{1}{3}$ pour une distribution ternaire donnée par l'auteur. Pour une aire convexe l'auteur montre, en s'appuyant sur un résultat antérieur [J. London Math. Soc. 14, 262–272 (1939); ces Rev. 1, 158] que la borne inférieure peut être améliorée et portée à 4/9, cette borne étant atteinte pour les régions triangulaires. Signalons encore que l'auteur établit en passant l'unicité du point extrêmal lorsque la distribution est connexe.

E. Blanc (Clermont-Ferrand).

Fiala, Félix. Sur les polyèdres à faces triangulaires. Comment. Math. Helv. 19, 83–90 (1946).

In certain isoperimetric inequalities of differential geometry the principal geometric notions involved are length, area, and total curvature. Analogous inequalities hold for polyhedral surfaces with triangular faces provided these geometric entities are suitably defined. Previous results have been obtained in this direction by C. Blanc [same Comment. 13, 54–67 (1940); these Rev. 2, 293]. For a simply-connected open polyhedral surface with triangular faces, the curves C considered consist of edges; the length L of C is the number of edges of which C is composed; the area A bounded by a closed curve C is the number of faces enclosed by C; the curvature at a vertex P is $6-\lambda(P)$, where $\lambda(P)$ is the number of edges abutting at P. It is shown, for instance, that for closed curves on surfaces of nonpositive curvature we have $L^2\geqq 6A$, while on surfaces of negative curvature the inequality can be strengthened to $L^2\geqq 7A$.

E. F. Beckenbach (Los Angeles, Calif.).

NUMERICAL AND GRAPHICAL METHODS

★**Gloden, A. Table des Bicarrés X^4 pour $1000 < X \leq 3000$.** A. Gloden, Luxembourg, 1946. 17 pp.

Neuschuler, L. On tabulating a class of inexplicit functions of four variables. C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 461–464 (1945). [MF 16647]

This paper describes a method for tabulating $v=f(x, y, z, u)$ defined by the equation $\phi[\eta(v, x), y]=\psi[\mu(v, z), u]$, where $\mu(v, z)/\eta(v, x)=\gamma(x, z)$ and does not depend on v. [This implies $\eta(v, x)=f_1(v)f_2(x)$, $\mu(v, z)=f_1(v)f_3(z)$, so that $\log\gamma(x, z)=\log f_2(x)-\log f_3(z)$; that is, (D) (see below) may consist of two tables of single entry.] The paper depends on the construction of four auxiliary tables of double entry, giving (A) values of v such that $\eta(v, x_i)=\alpha_1(1+\epsilon)^{k-1}$, (B) values of $\phi[\alpha_1(1+\epsilon)^{k-1}, y_j]$, (C) values of $\psi[\alpha_1(1+\epsilon)^{r-1}, z_l]$, (D) values of $\log_{1+\epsilon}\gamma(x_i, z_l)\equiv N$. The tables are at unit interval in l and r, for appropriate values of x, y and z. Given x_i, z_l, the value N is read from table (D). This value N is the difference between the values of k and r in tables (B) and (C) which are then placed with columns y_j, z_l ranged alongside one another accordingly, so that equal entries may be located and the value of k read off; this yields v from table (A).

The modification of the method applicable when $\mu(v, z)-\eta(v, x)=\gamma(x, z)$ is also discussed; in this case the values $\alpha_1(1+\epsilon)^{k-1}$, in geometrical progression, are replaced by numbers $\alpha_1+\epsilon(k-1)$ in arithmetical progression. The case when $F\{\eta(v, x), \mu(v, z)\}=\gamma(x, z)$ and is independent of v is also discussed. *J. C. P. Miller* (London).

Neĭšuler, L. The tabulation of functions. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 1157–1176 (1946). (Russian)

Expository lecture.

Akušskiĭ, I. Ya. A brief description of punched-card machines. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 1081–1120 (1946). (Russian)

Lyusternik, L. Some problems of computational mathematics. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 1147–1156 (1946). (Russian)

Expository lecture.

Gutenmaher, L. I. Electrical models (analogies) of physical phenomena and some of their applications in technology and physics. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 1121–1146 (1946). (Russian)

Byhovskiĭ, M. L. The new Bush differential analyzer. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 1177–1198 (1946). (Russian)

Cf. Bush and Caldwell, J. Franklin Inst. **240**, 255–326 (1945); these Rev. **7**, 339.

Comrie, L. J. The application of commercial calculating machines to scientific computing. Math. Tables and Other Aids to Computation **2**, 149–159 (1946).

Artobolevskiĭ, I. I. On some mechanisms for drawing lines. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 963–968 (1946). (Russian)

The intersection of the long bars of a crossed parallelogram (anti-parallelogram) linkage traces an ellipse if one of the short bars is held fixed. Two short blocks sliding on the long bars and pivoted together can carry a scriber or cutting tool. This arrangement can be used as a device for machine tools to replace more complicated gearing arrangements. Similar arrangements are shown for describing hyperbolas and parabolas. By the use of such devices, in combination with rigid right angle bars and rigid double slide blocks, a line is moved so that its envelope is a conic section. This allows the use of a milling cutter or a grindstone for shaping conic sections in a machine tool.

The roulettes of the conics can be traced by removing the fixation of the fixed bar and placing nonslipping wheels on the crossed bars in the manner described by R. C. Yates [Amer. Math. Monthly **38**, 573–574 (1931)]. *M. Goldberg* (Washington, D. C.).

Reiter, R. Das Spiegeltangentometer und seine Anwendung zur Bestimmung des Differentialquotienten von Kurven. Z. Instrumentenkunde **63**, 424–426 (1943).

A description of the well-known "tangent meter," a device with a mirror [prisms are also used] mounted so as to rotate at the center of a graduated circle, the amount of rotation being adjusted manually until the given curve and its image cease to show a kink. *P. W. Ketchum.*

Roma, Maria Sofia. Il metodo dell'ortogonalizzazione per la risoluzione numerica dei sistemi di equazioni lineari algebriche. Ricerca Sci. **16**, 309–312 (1946).

It is proposed to solve systems of linear algebraic equations by replacing the original system by an equivalent orthogonal system. Computational details are to appear later. *J. W. Tukey* (Princeton, N. J.).

Hallert, Bertil. Über einige Verfahren zur Lösung von Normalgleichungen. Z. Vermessungswesen **72**, 238–244 (1943).

The author explains a method of solving the normal equations associated with a set of observation equations in the theory of errors. The virtue of the method lies in the fact that it is possible at the same time to calculate the weight coefficients (which are found from the principal diagonal of the adjoint matrix) without having to calculate all the non-diagonal elements of the matrix. *W. E. Milne.*

Guttman, Louis. Enlargement methods for computing the inverse matrix. Ann. Math. Statistics **17**, 336–343 (1946).

The enlargement method is the author's name for the method of providing techniques for inverting any nonsingular matrix by building the inverse out of the inverses of successively larger submatrices. He overlooks the fact that the condensation method, the enlargement method and the basic formula itself have been extensively described by T. Banachiewicz [especially, Acta Astronomica. Sér. C. **3**, 42–67 (1937), in particular, pp. 51, 54–56; cf. Zentralblatt für Math. **17**, 416 (1938)] earlier than all the author's references. However, even Banachiewicz merely rediscovered independently a relation given by Schur [J. Reine Angew. Math. **147**, 205–232 (1917), in particular, p. 217]. The two basic relations

$$\text{(I)}\quad \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} P & 0 \\ R & E \end{pmatrix} \begin{pmatrix} E & P^{-1}Q \\ 0 & S_1 \end{pmatrix}, \quad S_1 = S - RP^{-1}Q,$$

so that

$$\text{(Ia)}\qquad \det\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \det P\cdot\det S_1,$$

$$\text{(II)}\qquad \begin{pmatrix} P & Q \\ R & S \end{pmatrix}^{-1} = \begin{pmatrix} P^{-1}+TU & -T \\ -S_1^{-1}U & S_1^{-1} \end{pmatrix},$$
$$T=P^{-1}QS_1^{-1},\quad U=RP^{-1},$$

should be called "Schur's relations." *E. Bodewig.*

★Johansen, Paul. A formula for osculatory interpolation. Festskrift til Professor, Dr. Phil J. F. Steffensen fra Kolleger og Elever paa hans 70 Aars Fødselsdag 28. Februar 1943, pp. 76–81. Den Danske Aktuarforening, Copenhagen, 1943. (Danish)

Après avoir donné en écriture symbolique une formule d'interpolation de Birkhoff [Trans. Amer. Math. Soc. 7, 107–136 (1906)] l'auteur y apporte des précisions dans le cas où l'on se propose de déterminer un polynome $P(x)$, de degré n, tel que $P^{(\nu)}(x_\nu)=f^{(\nu)}(x_\nu)$, $0\leqq\nu\leqq n$, où f désigne la fonction à interpoler. Il trouve

$$f(x)=P(x)+R(x)=\sum_{\nu=0}^{n}\varphi_\nu(x)f^{(\nu)}(x_\nu)+R(x),$$

avec $R(x)=\varphi_{n+1}(x)f^{(n+1)}(\xi)$, où les φ_ν sont des polynomes en x, qui ne dépendent que des x_ν, pour lesquels une écriture symbolique est donnée, et où ξ est un nombre compris entre le plus petit et le plus grand des nombres $(x; x_0, x_1, \cdots, x_n)$.
J. Favard (Paris).

Hummel, P. M. The accuracy of linear interpolation. Amer. Math. Monthly 53, 364–366 (1946).

The error of linear interpolation in an interval (a, b) in which $f''(x)$ and $f'''(x)$ are continuous and have constant signs is shown to lie between $(b-x)(x-a)k_1$ and $(b-x)(x-a)k_2$, where $k_1=\{f(b)-f(a)-(b-a)f'(a)\}/(b-a)^2$, and k_2 is a similar expression with a and b interchanged.
P. W. Ketchum (Urbana, Ill.).

★Bjerreskov, S. On some applications of symbolic computation in the theory of interpolation. Festskrift til Professor, Dr. Phil. J. F. Steffensen fra Kolleger og Elever paa hans 70 Aars Fødselsdag 28. Februar 1943, pp. 24–28. Den Danske Aktuarforening, Copenhagen, 1943. (Danish)

Exposé d'une voie symbolique élémentaire pour l'obtention de la formule de Lagrange, des formules de quadrature qui s'en déduisent, et enfin de la formule d'Euler.
J. Favard (Paris).

Jones, C. W., Miller, J. C. P., Conn, J. F. C., and Pankhurst, R. C. Tables of Chebyshev polynomials. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 187–203 (1946).

As stated by the authors, the object of this paper is twofold: first, to present a table of the Chebyshev polynomials $C_n(x)=2\cos(n\cos^{-1}\frac{1}{2}x)$ for $n=1(1)12$ and $x=0(0.02)2.00$, values being exact or to 10 decimals; second, to provide a working list of coefficients and formulae relating to these and allied functions. *T. N. E. Greville.*

Miller, J. C. P. Two numerical applications of Chebyshev polynomials. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 204–210 (1946).

Computational processes are outlined, with numerical examples, in which the table of Chebyshev polynomials given in the preceding paper may be used with effect in interpolation and Fourier synthesis. *T. N. E. Greville.*

Salzer, Herbert E. Coefficients for facilitating the use of the Gaussian quadrature formula. J. Math. Phys. Mass. Inst. Tech. 25, 244–246 (1946).

The practical use of the Gaussian quadrature formulas is complicated by the fact that the ordinates have to be calculated at unequally spaced and (usually) irrational values of the abscissas. Since errors in the calculated ordinates cannot be detected by simple differences the author proposes the use of divided differences. These too are inconvenient to calculate. However, since each divided difference is a linear combination of ordinates with coefficients which are functions of the abscissas alone, and since for n points the abscissas are always the same numbers (roots of $P_n(x)=0$), it is possible to calculate the coefficients once for all for each n. The paper includes a table of the coefficients for the $(n-1)$th divided differences for all values of n from $n=3$ to $n=10$ inclusive. *W. E. Milne* (Corvallis, Ore.).

Salzer, Herbert E. Table of coefficients for double quadrature without differences, for integrating second order differential equations. J. Math. Phys. Mass. Inst. Tech. 24, 135–140 (1945). [MF 15128]

If numerical values of f_i'' of a function $f''(x)$ are given at equidistant arguments $x_i=x_0+ih$ Lagrange's formula gives the polynomial of nth degree passing through any set of $n+1$ consecutive points (x_i, f_i''). The second integral $f(x)$ of $f''(x)$ is, under certain conditions, numerically identical with the second integral of these polynomials and its value f at $x=x_p$ is therefore given by $f_p\doteq h^2\sum C_i(p)f_i''+A^px_p+B$, where the summation extends over the $n+1$ points chosen and the $C_i(p)$ depend on the position of x relative to the x_i. Tables of $C_i(p)$ are given in the form $D_i^p(p)/D(p)$ with integer D's for $n=2(1)10$ and $x_p=x_i$, $i=1, 2, \cdots, n+1$. The advocated method of solving equations of the type $y''=\phi(x, y)$ with the help of these tables is similar to the "Milne method," but finite difference formulae are often preferable. *H. O. Hartley* (London).

Pascal, Ernesto. Sull'integrazione meccanica delle equazioni differenziali, e in particolare di quella lineare di 2° ordine ausiliaria dell'altra non lineare che è fondamentale per la fisica atomica. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 11, 209–243 (1941).

A discussion is given of the theory of the integraph of Abdank-Abakanowicz with various improvements and modifications which the author has made in order to enable him to solve special types of first and second order differential equations, such as the general linear first order equation, the equation $y''=(y^3/x)^{\frac{1}{2}}$, etc. The instruments are more compact but much less flexible than a differential analyser. The input unit for x (a two-wheeled carriage which rolls parallel to the x-axis and which carries the entire instrument), the input and output units for the dependent variables (small carriages carrying pointers and pencils, respectively, which roll on tracks attached to the main carriage), and the integrator (a friction wheel which constrains the motion of the output pencil) are retained in substantially the form used in the original integraph. The modifications consist mainly in the different types of linkages which are used to connect the integrator and the input and output

units. These linkages involve curved tracks, and cams, as well as parallelograms. *P. W. Ketchum* (Urbana, Ill.).

Vietoris, L. Eine Fehlerquelle bei den Führungsrädern von Integraphen. Z. Instrumentenkunde **64**, 123–129 (1944). [MF 13629]

The author investigates inaccuracies occurring in an integraph and in tests (similar to the "circle test" for the differential analyser) with the "Ott integraph." He has determined empirical "misclosure values." These are the differences in the readings at beginning and end of an integrating run along the closed contour of an oblong figure extending in the direction α against the main axis of the instrument. Plotting of these misclosure values against α suggests that they are caused by an outward tilt of the integrating wheel when following sharply curved contours. An asymmetry of the plot with regard to positive and negative α could not be explained in this way. The type of wheel used had only minor effects on the plots.

H. O. Hartley (London).

Berger, Erich Rud. Bestimmung von Deviationsmomenten mit dem Trägheitsmomenten-Planimeter. Z. Instrumentenkunde **61**, 381–384 (1941).

The product of inertia $I_{xy}=\int xydA$ (deviation moment or centrifugal moment) can be obtained by the use of a moment planimeter which measures the moments about the two axes passing through the origin but inclined at an angle α to the x-axis. For example, taking α equal to 45°, $I_{xy}=\frac{1}{2}(I_{-\alpha}-I_{+\alpha})$. The most probable location of the center of gravity of a plane figure is given by x, y, where $x=(M_- - M_+)/2^{\frac{1}{2}}F$ and $y=\{2M_0+(M_-+M_+)/2^{\frac{1}{2}}\}/3F$; here F is the measured area and M_0, I_0, M_-, I_-, M_+, I_+ are the measured moments and moments of inertia about an arbitrary axis through an arbitrary point and about two other lines through the same point but inclined at 45° to the first axis. The inclinations, τ and $\tau-90°$, of the principal axes to the x-axis are obtained from

$$\tan 2\tau=F(I_- - I_+)\\ -(M_-^2-M_+^2)/\{F(I_-+I_+-2I_0)-(M_-^2+M_+^2-2M_0^2)\}.$$

If F is the area of the half meridian section of a solid of revolution, M_0 and I_0 are its moment and moment of inertia about the edge which is the axis of revolution and I_{xy} is the product of inertia as determined in the above manner using an arbitrary y-axis, then the volume is given by $V=2\pi M_0$, the moment about the y-axis is $M_y=2\pi I_{xy}$, $M_0=(M_-+M_+)/2^{\frac{1}{2}}$, and the location of the center of gravity is given by $x=M_y/V=(I_- - I_+)/2^{\frac{1}{2}}(M_-+M_+)$.

M. Goldberg (Washington, D. C.).

Zech, Theodor. Über das Sprungstellenverfahren zur harmonischen Analyse. Arch. Elektrotechnik **36**, 322–328 (1942).

The process in question replaces the integral formula for Fourier coefficients by the formula obtained by repeated integration by parts, thus expressing the coefficients formally as series involving the jumps of the function and its derivatives. Uncritical applications of formal evaluations of these series have led to contradictions. The author traces the difficulties to the fact that $\sin n\pi=0$ only if n is an integer and gives correct formulas. *R. P. Boas, Jr.*

Fouché, André. Sur la détermination immédiate de l'amplitude et de la phase de l'harmonique de rang n d'une fonction de période 2π, présentant, ainsi que ses dérivées successives, un nombre fini de sauts. Application pratique au cas des cames et des ressorts de rappel en vue d'éviter les phénomènes de résonance. C. R. Acad. Sci. Paris **223**, 779–780 (1946).

The author rediscovers a known method for calculating the Fourier coefficients of a function made up of a finite number of polynomial pieces. [Cf. the preceding review.]

R. P. Boas, Jr. (Providence, R. I.).

Blume, Hans. Über die Analyse kurzer Kurvenzüge. Z. Angew. Math. Mech. **23**, 346–358 (1943).

Stumpff [Grundlagen und Methoden der Periodenforschung, Springer, Berlin, 1937] has given methods of harmonic analysis for the determination of frequencies, amplitudes, and phase angles of pure sine wave components of empirical curves. His analysis is limited to "long" pieces of the empirical function. The object of the present paper is to weaken this restriction. In so doing it is also possible to find components which are exponentially-damped sine waves. More specifically, the analysis interval (otherwise at one's disposal) is assumed to be longer than the given interval of definition of the function, which latter must be at least three times as long as a wave length of any component. Following Stumpff, the author uses the periodogram as the essential tool. This is the curve swept out in the complex plane by the complex Fourier coefficients c_n as the index n varies continuously as well as through integral values. These coefficients may be denoted by $c(x, q)$, where x replaces the discrete variable n and q is the left hand end point of the analysis interval $(q, q+p)$. The given function is assumed known in the interval $(p, p+l)$, where $l<p$, and is made identically zero elsewhere. Values of $c(x, q)$ for various x and q are calculated either numerically or graphically with the aid of a harmonic analyser. Amplitude and phase diagrams, in which $|c(x, q)|$ and $\arg (c(x, q))$, respectively, are plotted as functions of q for various fixed x's, may then be constructed. The frequency α of a component may be determined approximately by means of the property that the periodogram approaches a maximum absolute value when x is near α. A value $x=\alpha$ is also characterized in the phase diagram by the absence of discontinuities which otherwise occur at $q=l$ and $q=p$. If the phase diagram is plotted for a frequency x which is approximately equal to α, a more accurate value of α is obtained from measurements of the slope of the resulting curve (which will be nearly a straight line) in the interval from p to $p+l$. The damping constant, amplitude and phase angle may then be calculated easily from other measurements on these diagrams. Numerical examples are given to illustrate the method. The possibility of further generalizations is indicated for components with nonexponential damping.

P. W. Ketchum (Urbana, Ill.).

Callender, A. Simple differential equations arising in physics; rapid solution by using hatchet planimeters. J. Sci. Instruments **23**, 77–81 (1946). [MF 16326]

The author describes a simple integraph (the hatchet) for the approximate solution of differential equations of the type (1) $d\theta/dt=(\phi(t)-\theta)\tau^{-1}$, with τ constant, occurring, mainly, in heat-conduction problems. The "hatchet" (in its simplest form) consists of an arm of length L having, at one of its ends, a slightly curved knife (hatchet) and a

pointer at the other end. It is placed on the graph on which $\phi(t)$ is plotted and the pointer (held lightly) made to follow the ϕ-contour. The trailing hatchet then traces a curve satisfying $\sin(\tan^{-1} d\theta/dt) = \{\phi(t)-\theta(t)\}/L$. If $L \gg \mathrm{var}\,|\phi|$ we have approximate identity of sine and tangent and therefore an approximate plot of $d\theta/dt = (\phi-\theta)/L$. The error is discussed in a special case, but its accumulation, which might be serious, is not considered. A more elaborate construction of the hatchet is said to avoid the error altogether, but it would appear to be doubtful whether, in that form, the hatchet is better than the usual sharp-edged integrating wheel. Generalisations to systems of equations of type (1) are given when two (or more) hatchets are coupled.

H. O. Hartley (London).

Bergman, Stefan. Construction of a complete set of solutions of a linear partial differential equation in two variables, by use of punch card machines. Quart. Appl. Math. **4**, 233–245 (1946).

In an earlier paper [Duke Math. J. **6**, 537–561 (1940); these Rev. **2**, 201] the author developed a procedure for the determination of a function U satisfying a second order linear partial differential equation in two variables in a domain D and assuming prescribed values on the boundary of D. One of the essential steps is the determination of a sequence of functions $Q^{(\nu)}(x, y)$ satisfying a recurrence formula. In the present paper it is assumed that the coefficients in the differential equation are polynomials, in which case the $Q^{(\nu)}(x, y)$ are also polynomials with coefficients $q^{(\nu)}_{m,n}$. These $q^{(\nu)}_{m,n}$ are expressible as linear combinations of the $q^{(\nu-1)}_{i,j}$ with coefficients which are themselves combinations (not linear) of the constant coefficients in the polynomial coefficients of the differential equation. The calculation of the $q^{(\nu)}_{m,n}$ is laborious at best. The purpose of the present paper is to exhibit a procedure adapting the computation to punch card machines by the use of cards and stencils. The coefficients of the $q^{(\nu-1)}_{i,j}$ are entered on the cards. The stencils are cards with holes cut in them; the quantities $q^{(\nu-1)}_{i,j}$ are entered on the stencils in such a way that when the stencil is placed over the card properly, quantities on the cards appear in the holes in the stencil beside the $q^{(\nu-1)}_{i,j}$ by which they are to be multiplied. The cards and stencils are illustrated by figures with detailed explanations. The paper concludes with a numerical example.

W. E. Milne (Corvallis, Ore.).

Frocht, Max M. The numerical solution of Laplace's equation in composite rectangular areas. J. Appl. Phys. **17**, 730–742 (1946).

The method applies to rectangular areas or to areas like angles, channels, etc., which are composed of rectangles. A network of lattice points is chosen at which the values of a harmonic function, assumed given on the boundary, are to be determined. For an $m\times(n+\delta)$ rectangle, $0<\delta\leqq 1$, the lattice points are spaced to give mn square subdivisions and m partial squares along the right hand side. The usual approximation is assumed for the value of a harmonic function at the center of a square or partial square as a linear combination of the values at the corners. This approximation is applied "diagonally" as well as in the normal way. The problem is to reduce the computation to a minimum by making use of the combinatorial properties of the lattice. It is sufficient to compute the values of the function at a set of "key points" such that every point of the lattice will be the center of a square or partial square whose corners are either key points or boundary points. In order to find these key values the author makes use of chains of special subsets of the lattice. Attention is restricted to cases where $m=3$ or 4. In such cases the subsets or "links" of the chains consist of rectangular $m\times(n+\delta)$ lattices with $n=2$ or 3. There is just one key value in a link; expressions for it are obtained in terms of the boundary values around the link. When the link is in the middle of a chain, however, two of these boundary values are key values of adjacent links and hence unknown. A chain of k links, starting with a link adjacent to the left or right end of the given rectangle, gives rise to a linear recurrence relation between the kth and $(k+1)$th key values in the chain. When two chains of this sort extend across the given rectangle and meet, the two recurrence relations for the two chains can be solved explicitly for the key value where they meet. The expression thus obtained is a linear combination of "Laplacian perimeters" P_i. The latter are defined as linear combinations of the known values on the boundary of the ith link.

To apply the method to a rectangle, the computer first calculates the P_i from the given boundary values and then finds the key values by forming linear combinations of the P_i, the coefficients of which are found directly from tables given in the paper. For an angle or other composite figure, the procedure requires the solution of simultaneous linear equations to determine the key values at the junctions of the rectangles which make up the area.

P. W. Ketchum (Urbana, Ill.).

Wood, H. W. Nomograms for some astronomical computations. J. Proc. Roy. Soc. New South Wales **79**, 153–159 (1946).

Alignment charts are constructed for computations occurring in connection with Kepler's equation, refraction and solution of spherical triangles. *E. Lukacs.*

Barracco, E. Modificazioni di una formula per il calcolo dei corsi teorici dei titoli a reddito fisso, nel caso in cui si tenga conto delle imposte e tasse. Giorn. Ist. Ital. Attuari **12**, 43–52 (1941).

Sibirani, F. Sugli ammortamenti continui. Giorn. Ist. Ital. Attuari **12**, 1–13 (1941).

It is assumed that the force of interest is a function $\delta(t_0, t)$ of the time t as well as of the initial moment t_0 of the capitalization. Five agreements concerning the amortization of a debt by a variable continuous annuity are discussed. Essentially, these may be reduced to the following three cases. (1) At the moment the loan is taken the present value of the annuity should be equal to the debt. (2) At the moment the debt is paid up the amount of the annuity should be equal to the amount of the debt. (3) At each moment the annuity is used partly to reduce the indebtedness and partly to pay interest. It is shown that these agreements are equivalent if and only if the law of capitalization is multiplicative. A law of capitalization is said to be multiplicative if $m(x, z) = m(x, y)\cdot m(y, z)$. In this formula $m(x, y)$ denotes the amount at time y of a monetary unit invested at time x. *E. Lukacs* (Cincinnati, Ohio).

Marseguerra, V. Sulle tavole di mutualità che portano ad uno stesso capitale accumulato. Giorn. Ist. Ital. Attuari **12**, 227–229 (1941).

Correction and amplification of a paper in the same Giorn. **8**, 230–246 (1937).

Niedermann, Hans. Untersuchungen über den Wahrscheinlichkeitscharakter der Sterblichkeit. Mitt. Verein. Schweiz. Versich.-Math. 46, 131–166 (1946). [MF 16690]

The author discusses in what degree mortality follows the laws of probability. The annual death rates $q=q(x, t)$ are considered as functions of age x and time t and the fluctuations compared with those expected from the urn scheme. As to the x-direction, the difficulty is that the result depends on the chosen graduated table. This difficulty can be avoided by using the variate difference method of graduation [see, for example, "Student," Biometrika 10, 179–180 (1914)]. *P. Johansen* (Copenhagen).

RELATIVITY

Hill, E. L. A note on the relativistic problem of uniform rotation. Phys. Rev. (2) 69, 488–491 (1946). [MF 16527]

Le problème relativiste du disque tournant a été envisagé jusqu'ici au point de vue de la définition classique d'une rotation uniforme, d'après laquelle chaque point mobile du disque se meut avec une vitesse qui est une fonction linéaire du rayon de sa trajectoire. Les auteurs qui se sont occupés de la question n'ont pas remarqué que cela impose une condition restrictive aux dimensions du disque parce que, si l'on admet cette loi de vitesse aussi grandes que soient les distances de l'axe de rotation, on peut excéder la vitesse c de la lumière, ce qui est contraire aux principes fondamentaux de la relativité. L'auteur propose une nouvelle définition de la rotation uniforme de sorte que la vitesse soit quasi linéaire aux petites distances de l'axe et tend vers la vitesse c lorsque la distance croit indéfiniment.

Il définit la rotation uniforme comme le mouvement d'un fluide tel que la vitesse angulaire d'une portion infiniment petite autour de chaque point P du fluide, calculée dans un système d'axes par rapport auquel P est à l'instant au repos, ait une valeur déterminée ω_0 indépendante du choix de P. L'application des principes de la relativité restreinte conduit à une équation de Riccati pour la vitesse inconnue $v(R)$ en fonction de la distance R du point P de l'axe de rotation, $dv/dR+2\omega_0c^{-2}v^2+v/R-2\omega_0=0$. Si l'on pose $z=2i\omega_0R/c$ et $v(R)=cw'(z)/w(z)$ on trouve l'équation de Bessel $w''+z^{-1}w'+w=0$ et la solution cherchée est donnée par $v(R)=-icJ_1(2i\omega_0R/c)/J_0(2i\omega_0R/c)$, où J_0 et J_1 sont les fonctions de Bessel d'ordre zéro et un. Cette fonction satisfait bien aux conditions imposées parce que si R est très petit on a $v(R)=\omega_0R$ et d'autre part on a $\lim_{R\to\infty} v(R)=c$ [cf., par exemple, J. Hadamard, Cours d'Analyse, tome 2, Hermann, Paris, 1930, pp. 402–403]. A la fin de la note l'auteur exprime l'opinion que le paradoxe d'Ehrenfest dans le problème du disque tournant et la question de décider la géométrie du mouvement, au sens de la relativité générale [cf. A. Einstein et L. Infeld, The Evolution of Physics, Simon and Schuster, New York, 1942, p. 240] peuvent être éclairées seulement par une théorie de la génération du mouvement de rotation. *G. Lampariello* (Messina).

Rosen, Nathan. Note on the problem of uniform rotation. Phys. Rev. (2) 70, 93–94 (1946).

Comment on the paper reviewed above.

Van Mieghem, Jacques. Les ondes gravifiques et les variables de Th. De Donder. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 30 (1944), 291–297 (1945).

Cauchy's initial value problem for Einstein's gravitational equations is considered. For arbitrary coordinates the equation of the characteristics is claimed to be an identity. New coordinates, due to De Donder, are introduced; these are essentially four independent solutions of D'Alembert's equation. The characteristic equations of gravitational waves are obtained in terms of De Donder's coordinates. *A. E. Schild* (Pittsburgh, Pa.).

Van Mieghem, Jacques. Les ondes du champ gravifique-électromagnétique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 30 (1944), 397–404 (1946).

The characteristics of combined electromagnetic and gravitational field equations are obtained in terms of De Donder's coordinates [see the preceding review] and of electromagnetic potentials which satisfy the Lorentz condition of vanishing divergence. *A. E. Schild.*

Van Mieghem, Jacques. Les ondes gravifiques et les variables de Th. De Donder. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 30 (1944), 410–413 (1946).

[Cf. the second preceding review.] The characteristic equations of gravitational waves are obtained in arbitrary coordinates. *A. E. Schild* (Pittsburgh, Pa.).

Lifshitz, E. On the gravitational stability of the expanding universe. Acad. Sci. USSR. J. Phys. 10, 116–129 (1946).

The metric of the expanding universe of general relativity is of the form $ds^2=-c^2dt^2+a(t)^2dl^2$, where

$$dl^2=d\chi^2+\sin^2\chi(\sin^2\theta d\phi^2+d\theta^2)$$

for the closed model, with a corresponding form for the open model. (The "parabolic" model is not discussed here.) The stability of this universe is examined by considering those small perturbations which correspond to small deformations of the spatial part $a(t)^2dl^2$. This deformation is arbitrary except that pressure is assumed to remain isotropic. Writing h_{ij} for the variation δg_{ij} in the fundamental tensor of space-time, h_{0i} is assumed to be zero, and $h_{\alpha\beta}$, $\alpha, \beta=1, 2, 3$, is a tensor under spatial transformations. The general equations connecting $h_{\alpha\beta}$, their derivatives and the variations of density and pressure are obtained and expressed in tensor form, referring to the 3-space with metric dl^2.

Subsequent discussion is simplified by the introduction of surface harmonics in flat 4-space, analogous to spherical harmonics in Euclidean space. The tensor $h_{\alpha\beta}$ is expressible in terms of surface harmonics of three kinds, scalar, vector and tensor, and each kind gives rise to a particular form of perturbation. From scalar harmonics there arise perturbations accompanied by changes of density, vector harmonics give rise to rotational perturbations and the perturbations corresponding to tensor harmonics are gravitational waves. In all cases the perturbations ultimately decrease in time and the expanding universe is therefore stable under the assumed conditions. *A. G. Walker* (Liverpool).

Lifshitz, E. On the gravitational stability of the expanding universe. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 16, 587–602 (1946). (Russian. English summary)

An English translation is reviewed above.

Jaiswal, J. P. On the electric potential of a single electron in gravitational fields. I. Proc. Benares Math. Soc. (N.S.) 7, 17–25 (1945).

The electrostatic potential of a single electron in a gravitational field in general relativity theory is obtained for the

metric

$$ds^2=(1-2m/r)dt-c^{-2}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)(1+2m/r).$$

The present result differs from those given earlier by E. T. Whittaker [Proc. Roy. Soc. London. Ser. (1) **116**, 720–735 (1927)] in that the radial solution cannot be expressed in terms of known functions. *C. Kikuchi.*

Milne, E. A. On the conservation of momentum. Proc. Roy. Soc. London. Ser. A. **186**, 432–442 (1946).

Equations corresponding to the classical integrals of linear and angular momentum are obtained from Milne's equations of motion for a set of particles under mutual gravitational attraction. *A. E. Schild* (Pittsburgh, Pa.).

Infeld, L., and Schild, A. E. A new approach to kinematic cosmology. II. Phys. Rev. (2) **70**, 410–425 (1946).

[For part I, see the same Rev. (2) **68**, 250–272 (1945); these Rev. **7**, 397.] Maxwell's equations, Lorentz's ponderomotive equations and Dirac's equations for the electron are investigated in space-times in which

$$ds^2=\gamma(t, r)(dt^2-dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2),$$

where γ can have one or other of the forms

(I) $\gamma=(1-a)^{-2}f\{t/(1-a)\}$, (II) $\gamma=\gamma(a)$, (III) $\gamma=\gamma(t)$, and $a=t^2-r^2$. The theory differs from general relativity in that no use is made of Einstein's gravitational equations. Hence it can be shown that solutions of Maxwell's or Dirac's equations valid in flat space-time are also valid in open universes of type (II) or (III). In closed universes of type (I), each point has an image and this imposes a boundary condition on the solutions. In such universes the possible frequencies of free radiation are restricted and, for the free electron, there is a connection between the atomic constants occurring in Dirac's equation and the "cosmological" function γ. *G. C. McVittie* (London).

Roy, S. K. Certain inconsistencies in the mathematical theory of a new relativity of Dr. Sir Shah Sulaiman. Proc. Nat. Acad. Sci. India. Sect. A. **10**, 1–13 (1940).

Some errors and confusions in Sulaiman's theory of relativity are pointed out. In particular, it is proved that the theory gives a value for the advance of the perihelion of Mercury which does not fit the facts. *G. C. McVittie.*

Sulaiman, Shah. Reply to the preceding paper. Proc. Nat. Acad. Sci. India. Sect. A. **10**, 14–18 (1940).

[Cf. the preceding review.] The author refuses to accept S. K. Roy's criticisms. *G. C. McVittie* (London).

ASTRONOMY

Garavito Armero, Julio. Definitive formulas for the calculation of the motion of the moon by the Hill-Brown method, and with the notation used by Henri Poincaré in volume III of his Course in Celestial Mechanics. Revista Acad. Colombiana Ci. Exact. Fis. Nat. **6**, 560–570 (1946). (Spanish)

H. Poincaré [Leçons de Mécanique Céleste, vol. 2, part 2, Paris, 1909; the title erroneously refers to vol. 3] introduced an elegant and direct method for obtaining a literal development in terms of the parameter m of the coefficients arising in the series that represent Hill's variation orbit of the lunar theory. Poincaré limited his presentation to a study of the analytical features of the procedure. The details of the developments are contained in this paper, to the extent that the author derives the differential equations, particular solutions of which must furnish the contributions having successively $m^0, m^2, m^4, \cdots, m^{12}$ as factors. The solution is carried out only to the terms having m^2 as a factor, but is to be continued in a later article. The results agree, as they must, with the expressions obtained originally by G. W. Hill by a different procedure [Amer. J. Math. **1**, 5–126, 129–147, 245–260 (1878); Collected Mathematical Works, vol. 1, Washington, 1905, pp. 284–335].
D. Brouwer (New Haven, Conn.).

✶Kopal, Zdeněk. An Introduction to the Study of Eclipsing Variables. Harvard Observatory Monograph no. 6. Harvard University Press, Cambridge, Mass., 1946. x+220 pp. $4.00.

This volume deals principally with the theory of the light variation of eclipsing variables. Although the practical side of the subject, the analysis of light curves, is not entirely ignored, this aspect is definitely in the background. After an introductory chapter, the "geometrical" chapters II to V may be said to deal with the first approximation. These chapters are to a large extent a critical examination of the theory applicable to systems in which the separation of the components is sufficiently large so that the secondary effects may be ignored. This field was developed originally by H. N. Russell, in part in collaboration with H. Shapley, in 1912.

The secondary effects are treated in the "dynamical" chapters VI to IX, the first of which is introductory, dealing with the theory of equilibrium of a rotating star as well as with that of the distribution of brightness over the surface of a distorted star. The mathematical theory of the light variation in close systems in which the effects of varying surface brightness, ellipticity and reflection are taken into account is presented in chapters VII and VIII. The author has made original contributions to this subject [for example, Proc. Amer. Philos. Soc. **85**, 399–431 (1942); Astrophys. J. **96**, 20–27 (1942); these Rev. **4**, 73], many of which are further elaborated. In the final chapter the application to the analysis of light curves is examined. The effective application of the theory will require tabulations of the "associated α-functions" introduced in the theory. Bibliographical notes at the end of each chapter are a valuable feature of the book. *D. Brouwer* (New Haven, Conn.).

✶Sémirot, P. Chocs et solutions périodiques dans le problème des trois corps. Thesis, University of Paris, 1943. 101 pp.

An imaginary collision occurs in the problem of three bodies if a mutual distance vanishes without the vanishing of all three projections of this distance on the coordinate axes. The first part of the thesis deals with the nature of the singularity in the solution at such collisions; for example, in the case of a simple collision at $t=0$ (where exactly one mutual distance vanishes) it is shown that the solution can be expanded in powers of $t^{\frac{1}{3}}$.

The second part applies the transformation of Sundman to Euler's problem of two fixed centers of gravitation; the instability of the periodic solutions is established. In the third part the author maintains that Poincaré's periodic

solutions of the second kind [Les Méthodes Nouvelles de la Mécanique Céleste, vol. 3, Paris, 1899, pp. 362–371] do not exist when the Keplerian ellipses described by the masses $m_0=\epsilon M_0$, $m_1=\epsilon M_1$ about the finite mass m_2 for $\epsilon=0$ intersect and the masses m_0, m_1 collide at the point of intersection. *M. H. Martin* (College Park, Md.).

García, Godofredo. On the restricted problem of three bodies in the general theory of relativity. Actas Acad. Ci. Lima 9, 153–162 (1946). (Spanish)

The author considers briefly the differential equations of a restricted three-body problem with a potential-energy function derived from considerations of general relativity. *W. Kaplan* (Ann Arbor, Mich.).

García, Godofredo. Reduction of the equations of motion of three bodies of finite masses to the case in which one of the bodies has infinitely small mass. Actas Acad. Ci. Lima 9, 163–168 (1946). (Spanish)

The differential equations and their integrals for the general and restricted three-body problems are given in a unified form. *W. Kaplan* (Ann Arbor, Mich.).

Heckmann, O. Das statistische Gleichgewicht eines freien Systems von Massenpunkten. I. Z. Astrophys. 23, 31–56 (1944).

The standard procedure of statistical mechanics is to consider the equilibrium of enclosed, isolated systems of particles. It is then only necessary to take into account the energy integral for the canonical equations of the system. For the study of isolated stellar systems one must also consider the conditions of the conservation of the total momentum and angular momentum of the system. The probabilities for various configurations are calculated and the distribution function of the velocities corresponding to a given configuration is then derived. According to the customary procedure of stellar statistics this distribution function of the velocities is then described in terms of general streaming and peculiar motions. The streaming appears to be of such a nature that the central portion of the system rotates like a solid body, which, in the most general case of statistical equilibrium, is subject to precessional motion.

In the concluding section of the paper, the author shows that, in a system of constant angular momentum, there cannot exist a Maxwellian distribution of velocities with equal dispersion in all directions. It is furthermore found impossible for the entire system to rotate as a solid body. The author points out that deviations from rotation as a solid body and from a pure Maxwellian distribution should be most noticeable in moving clusters with relatively few members. A discussion of the resulting density laws and a comparison with results obtained by other statistical methods are reserved for a later paper. *B. J. Bok.*

Lindblad, Bertil. On the dynamics of stellar systems. Stockholms Observatoriums Annaler 13, no. 5, 32 pp. (1940). [MF 16904]

It is well known that, for stellar systems with an axis of symmetry, distribution functions of the form $f(I_1, I_2)$, where I_1 and I_2 are the energy and the angular momentum integrals, respectively, are compatible with Liouville's equation. However, the question of whether distributions of this form can be found which are also compatible with Poisson's equations has often been discussed inconclusively. The author reexamines this problem along the lines of an earlier investigation [Monthly Not. Roy. Astr. Soc. 98, 576–586 (1938)] and comes to the conclusion that such distributions exist and can be found. The method of argument is, however, too involved to be easily summarized.

S. Chandrasekhar (Williams Bay, Wis.).

Lindblad, Bertil. Remarks on the dynamical theory of spiral structure. Stockholms Observatoriums Annaler 14, no. 1, 32 pp. (1 plate) (1942). [MF 15542]

In this paper, which follows an earlier investigation [same Annaler 13, no. 10 (1941)] the author considers the effect of time-dependent perturbations on nearly circular orbits described in an axially symmetrical field in the plane $z=0$; the plane $z=0$ is also assumed to be a plane of symmetry. The orbits are referred in a rotating frame of reference to circular orbits which have the same constant of areas as the ones under consideration and the effect of perturbing potentials of the forms

$$(*)\qquad \varphi=C_0e^{\gamma t}\varpi^2\cos(\omega t+2\theta),$$
$$(**)\qquad \varphi=C_0e^{\gamma t}\varpi^2\cos(2\omega_0 t+2\theta)$$

are considered. Here ϖ and θ denote the cylindrical polar coordinates about the axis of symmetry, ω the mean angular velocity of the system, ω_0 the angular velocity in the circular orbit considered and C_0 and γ are constants. A disturbance of the form $(**)$ can result from the tidal effects of a neighboring mass. Solutions for the orbits in the rotating frame are obtained and examined particularly in cases in which the nearly circular orbits are on the verge of stability (that is, described in regions where the variation of the field of force approaches that of an inverse cube law). The variations of density induced in regions somewhat distant from the regions of instability of the circular orbits are also examined.

The bearing of the results of this analysis on the author's theory of the development of spiral structure in extragalactic nebulae is elaborated at length. *S. Chandrasekhar.*

Nobile, Vittorio. Sulla maniera di intendere e di trattare il problema della rotazione galattica. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 222–231 (1943).

Nobile, Vittorio. Sopra un gruppo di problemi astronomici connessi con quello della rotazione galattica.—La soluzione rigorosa in base ad un nuovo postulato. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 349–354 (1943).

Mineo, Corradino. Forma d'un pianeta dedotta dai valori della gravità in superficie. III. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 143–145 (1943).

The first two parts appeared in the same Rend. (6) 29, 529–535 (1939); (7) 1, 109–113 (1940); these Rev. 2, 25.

Lebedinsky, A. I. Structure of envelopes of novae. Astr. J. Soviet Union [Astr. Zhurnal] 23, 15–30 (1946). (English. Russian summary) [MF 16956]

The hydrodynamical equations of an extended spherical stellar atmosphere are investigated. These equations differ from the usual equations of gas dynamics in that in the equation of motion, the acceleration due to radiation pressure must be included and also the variation of temperature must be allowed for in accordance with the solution of radiative equilibrium [N. A. Kosirev, Monthly Not. Roy. Astr. Soc. 94, 430–443 (1934), equation (8); S. Chandrasekhar, ibid. 444–458 (1934), equation (51)].

The author first shows that under certain conditions the equations admit a "characteristic surface" moving with velocity $w=\pm(RT/\mu)^{\frac{1}{2}}$ (R the gas constant, μ the mean molecular weight, T the temperature). He next investigates the form of the solution in the neighborhood of such a characteristic surface and shows that the structure of such a "quasi-stationary" wave can be reduced to the solution z of an equation of the form $(z^2-z_0^2)\partial z/\partial y+4z^2+z^2-\beta z=0$, β=constant, which tends to z_0 as $y\to 0$. Here z is proportional to ρT^{-3} and $y=(T-T_0)/T_0$, where T_0 is the temperature on the characteristic surface. The equation is solved and the structure of envelopes of novae is interpreted in terms of the solutions. *S. Chandrasekhar.*

Severny, A. B. On the stability of gaseous stars. Astr. J. Soviet Union [Astr. Zhurnal] **23**, 137–140 (1946). (Russian. English summary)

Purely radial oscillations are considered for a star in radiative equilibrium throughout. The equation of oscillation is considered as an eigen-value equation and inequalities are obtained for the lowest period. Two crude sufficient conditions of stability for radial displacements are derived: one states that the ratio of specific heats must exceed $16G\rho_m^2a^2/(3\pi\bar{p})$ (ρ_m the maximum density, a the radius, $\bar{p}$ a mean pressure); the other, involving the ratio of gas pressure to total pressure, is satisfied whenever there is stability against convection. *T. G. Cowling* (Bangor).

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. XII. Astrophys. J. **104**, 191–202 (1946).

[Part XI appeared in the same vol., 101–132 (1946); these Rev. **8**, 59. This part is a continuation of part IX, same J. **103**, 165–192 (1946); these Rev. **7**, 489.] The problem of the diffuse reflection of radiation by a semi-infinite plane-parallel atmosphere is solved, allowance being made for the anisotropy of the scattered radiation. The phase function now considered is one which is expressible as a finite series of Legendre polynomials in the cosine of the angle of scattering. *G. C. McVittie* (London).

Alfvén, Hannes. On the cosmogony of the solar system. Stockholms Observatoriums Annaler **14**, no. 2, 33 pp. (1942). [MF 15541]

In this paper the effect of the general magnetic field of the sun is introduced as a major factor in the genesis of the solar system. The theory starts with the assumption that the sun in much the same condition as at present (except possibly for a faster rotation) enters a gaseous nebula consisting of neutral atoms with velocities corresponding to the temperature of interstellar space. These atoms fall in under the sun's gravitational field and acquire at a distance r a kinetic energy $E=GM_\odot Am_H/r$, where G is the constant of gravitation, $M_\odot$ the mass of the sun and Am_H the mass of the atom (in grams). When E becomes equal to the ionization energy eV of the atom, the atoms are assumed to get ionized. The distance r_i at which this is expected to happen is $r_i=GM_\odot Am_H/eV$. It is now argued that, since ions will be affected much more by the general magnetic field than by the gravitational field of the sun, the whole system of forces will be transformed at this distance r_i. Alfvén shows that, in general, the ions will be repelled, spiralling round the lines of force down the magnetic gradient until they reach the equatorial plane. At the same time, the matter is assumed to acquire a rotation on grounds explained in an earlier paper [Ark. Mat. Astr. Fys. **28A**, no. 6 (1942)]. For $A=7$ and $V=12$ volts, $r_i=8\times10^{13}$ cm., which is the radius of Jupiter's orbit. For matter of this composition, then, gravitation will first of all cause a uniform concentration on a spherical surface near the distance of the heaviest planet and then electromagnetic forces will spread this out into an equatorial disc, by projection along the lines of force, and cause it to rotate at the same time. The mass distribution on the disc is calculated and by apportioning to Jupiter the mass between the orbits of Jupiter and Saturn, to Saturn that between Saturn and Uranus and so on, Alfvén finds that the relative masses of the outer planets can be accounted for satisfactorily. Furthermore, he pictures the condensation of the matter in the equatorial disc into planetesimals and then planets somewhat along the lines suggested by Lindblad [Nature **135**, 133–135 (1935)].

If it is now assumed that the planets so formed are themselves magnetized, the primary process will operate on a reduced scale on the matter in the neighborhood of the planets and produce the satellites. With Jupiter the critical distance r_i is of the same order as the distance of the Galilean satellites. With Saturn r_i is less than the Roche limit and condensation could presumably not take place. In this way, Alfvén accounts for the ring system of Saturn. However, with Uranus and Neptune r_i is of the same order as the planetary radii themselves and no satellites would be expected. The primary process thus accounts for the outer planets and their inner satellites. For the terrestrial planets and the outer satellites of the outer planets a similar but somewhat different process is suggested. Alfvén suggests that in this case meteoric dust is involved. For, solid particles could presumably penetrate closer to the sun before they are volatilized and then ionized. It is this resulting gas repelled along the innermost lines of force that is finally assumed to form the terrestrial planets. Some of the gas ions reaching the outer planets are assumed to be responsible for the outer satellites on account of the high A/V values. *S. Chandrasekhar* (Williams Bay, Wis.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Carathéodory, C. Die Fehler höherer Ordnung der optischen Instrumente. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1943**, 199–216 (1944).

The author replaces the development of the characteristic function into a power series by a development according to certain polynomials which permits a simpler transformation if the coordinate origin is changed. To do so he introduces, instead of the coordinates x, y of the intersection point with the object plane and the direction cosines ξ, η the imaginary numbers $x\pm iy$, $\xi\pm i\eta$ in object and image space [cf. J. L. Synge, Amer. Math. Monthly **51**, 185–200 (1944); these Rev. **5**, 275, where a similar transformation was suggested]. Carathéodory develops the characteristic function as function of these coordinates with complex coefficients and investigates the members up to the fifth order. He gives only the general formulae for his theory. Only further detailed development will show the advantage of his formulae over the methods used hitherto. He claims that the use of

the new coordinates saves 75 per cent of the labor of calculation. *M. Herzberger* (Rochester, N. Y.).

Hufford, Mason E. Refractions by a thick lens which is equivalent to a compound lens system. Amer. J. Phys. 14, 259–266 (1946).

The familiar formulae of first-order optics are derived, using the assumption that a compound lens system can be explained by a single thick lens and assuming a formula for paraxial refraction at a single surface. *A. J. Kavanagh.*

Biot, A. Sur la correction de l'aberration sphérique dans les systèmes sphériques centrés. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **30** (1944), 34–39 (1945).

If the designer has available for the correction of spherical aberration a certain number of degrees of freedom in the specifications of a system (indices of refraction, radii, separations of surfaces), he can in principle correct this aberration for at least an equal number of rays. However, solution for the required specifications may yield imaginary values. The paper does not give any discussion of the manner in which the solution is to be obtained. *A. J. Kavanagh.*

Févrot, Ch. Caractéristique théorique d'un projecteur à miroir parabolique à source cylindrique axiale. Rev. Optique **23**, 261–276 (1944).

Zanotelli, Guglielmo. Assorbimento elementare della luce nel passaggio attraverso alle nubi. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 42–50 (1940).

This paper is concerned entirely with the absorption (conversion into heat) of light by a water drop in a cloud. The author first shows that nearly all the light incident on such a drop comes from other drops so distant that it may be treated as having plane wave-fronts. He then shows that the absorption is independent of the state of polarization of the incident light. Finally the energy absorbed by all the drops in a given volume of cloud is shown to depend only on the total liquid water content and not separately on the radius or number of the drops. The numerical results are claimed to be accurate within 1% for wavelengths between 0.45 μ and 0.81 μ, assuming that the drops are so large that the simple geometrical treatment of the paper is valid. The author points out that his value for the absorption is almost exactly $\frac{3}{4}$ of that stated without derivation by F. Albrecht [Meteor. Z. **50**, 478–486 (1933)]. It would be interesting to know the basis of the latter's statement.
W. E. K. Middleton (Ottawa, Ont.).

***Bouwkamp, Christoffel Jacob. Theoretische en Numerieke Behandeling van de Buiging door een Ronde Opening. [Theoretical and Numerical Treatment of the Diffraction through a Circular Aperture].** Dissertation, University of Groningen, 1941. 60 pp. (Dutch. English summary)

The exact solution of diffraction problems is known in very few cases, the only three-dimensional one being the diffraction by a sphere. The author considers only scalar waves with velocity potential φ, diffracted at a plane screen at which the boundary condition is either $\varphi_1=0$ or $d\varphi_2/dn=0$. For arbitrary form and number of openings he derives a modified Babinet principle according to which, if the disturbance caused by the opening is represented for the two boundary conditions by φ_1 or φ_2, respectively, then the disturbance caused by a screen of the form of the opening is, respectively, φ_2 or φ_1 in front of the screen, $-\varphi_2$ or $-\varphi_1$ behind it.

For the case of a circular opening (radius 1) these functions are then constructed by the method of eigenfunctions in spheroidal coordinates ξ, η, ψ for which

$$x=\xi\eta;\quad y, z=(1-\xi^2)^{\frac{1}{2}}(1+\eta^2)^{\frac{1}{2}}\cos\psi, \sin\psi,$$

by infinite series $\varphi=\sum c_nX_n(\xi)Y_n(\eta)$ with the differential equation

$$\frac{d}{d\xi}[(1-\xi^2)X_n']+(k^2\xi^2+\Lambda_n)X_n=0$$

for X_n, where $k=2\pi/\lambda$ is the wave number. In analogy with the method of Ince [Proc. Roy. Soc. Edinburgh **46**, 20–29 (1925); 316–322 (1926); **47**, 294–301 (1927)] for the Mathieu equation, the eigenvalues Λ_n are found as the roots of a continued fraction after developing X_n into a series of Legendre polynomials P_m. A second solution is found from the analogue of Neumann's relation between Q_m and P_m,

$$\Xi_n(\xi)=\tfrac{1}{2}e^{-ik\xi}\int_{-1}^{1}(\xi-t)^{-1}e^{ikt}X_n(t)dt,$$

and from this $Y_n(\eta)=\Xi_n(i\eta)$. A simple expression for the coefficients c_n is then found.

For $k<2$, power series are derived for numerical calculation. For $k^2=3, 4, \cdots, 10, 15, 20, 25, 50$ and 100 extensive tables of numerical results have been calculated directly from the continued fraction. The energy transmitted through the opening is given and compared with that according to the approximate Kirchhoff solutions. For small k these are quite different from the exact solutions, but for $k=10$ (diameter of opening $10/\pi$ wavelengths) the agreement is fair. The solutions with the first boundary condition represent the sound radiated by a vibrating circular disc.
F. Zernike (Groningen).

Copson, E. T. An integral-equation method of solving plane diffraction problems. Proc. Roy. Soc. London. Ser. A. 186, 100–118 (1946). [MF 16740]

According to Rayleigh [Theory of Sound, London, 1896, vol. 2, p. 107], any scalar wave function V in the half-space $z>0$, subject to certain boundary and regularity conditions, is completely determined by the values on the plane $z=0$ either of V or its normal derivative. A regular electromagnetic field in $z>0$ can then be expressed in terms of the boundary values of either E_x, E_y, H_z, or H_x, H_y, E_z, this duality corresponding with the invariance of Maxwell's equations under the transformation $\mathbf{E}\rightarrow\mathbf{H}$, $\mathbf{H}\rightarrow-\mathbf{E}$. The author gives the field components explicitly in terms of first order derivatives of certain wave functions, involving the tangential and normal components at the boundary. The two sets of formulae obtained are shown to be consistent with the general Larmor-Tedone vector solution of Maxwell's equations inside a closed surface [Baker and Copson, The Mathematical Theory of Huygens' Principle, Clarendon Press, Oxford, 1939].

The problem of electromagnetic reflection is treated when a perfectly conducting plane screen does not cover the whole plane $z=0$. The diffraction field is expressed in terms of first order derivatives of wave functions

$$u, v, w=(2\pi)^{-1}\int\int(e_x, e_y, h_z)\varphi dx'dy',$$

where the integration is over the holes S_1 in the screen (the dual problem is analogous). For the total (diffracted plus primary) field to be continuous through the hole, the ficti-

tious magnetic currents and charges e_x, e_y, h_z must satisfy one differential and three integral equations, involving the known values of the primary field on S_1. These equations are solved for Sommerfeld's [Math. Ann. **47**, 317–374 (1896)] well-known problem of diffraction by a half-plane in a manner different from that of Magnus [Z. Physik **117**, 168–179 (1941); these Rev. **4**, 32]. The diffraction by a small circular hole is solved by an analysis suggested by that of Bethe [Phys. Rev. (2) **66**, 163–182 (1944); these Rev. **6**, 165] for the same problem. The complementary problem of the diffraction by a small circular disc is treated similarly.

Finally, Babinet's principle in its rigorous form, suggested by unpublished work of H. G. Booker, is stated as follows. (i) Let the field $\mathbf{E}^i=\mathbf{F}$, $\mathbf{H}^i=\mathbf{G}$ be incident in $z>0$ on a perfectly conducting plane screen at $z=0$ (with holes S, metal S') and let $\mathbf{E}^1$, $\mathbf{H}^1$ be the total field in $z<0$. (ii) Let the complementary field $\mathbf{E}^i=-\mathbf{G}$, $\mathbf{H}^i=\mathbf{F}$ be incident in $z>0$ on the complementary screen (holes S', metal S) and let the total field in $z<0$ be $\mathbf{E}^2$, $\mathbf{H}^2$. Then the rigorous form of the principle is that $\mathbf{E}^1+\mathbf{H}^2=\mathbf{F}$, $\mathbf{H}^1-\mathbf{E}^2=\mathbf{G}$.

[Note by the reviewer. There remains only one question not properly accounted for by this analysis, namely whether or not line charges along the rim of the screen are necessary [cf. also the review of Bethe's paper cited above]. For instance, the proof of theorem 4 is incomplete. Suppose the integral equations (4.15)–(4.17) are solved rigorously, under the side-condition (4.14). It is not at once evident whether the wave functions u, v, w, defined by (4.12), fulfill (4.13). Now it can be shown that

$$-ikw-\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=(2\pi)^{-1}\int\int_{S_1}\left\{\frac{\partial}{\partial x'}(e_y\varphi)-\frac{\partial}{\partial y'}(e_x\varphi)\right\}dx'dy'.$$

Maxwell's equations are satisfied if and only if the right-hand integral vanishes. Because $e_x=E_x$, $e_y=E_y$ in the hole, the integral over S_1 can be transformed into an integral along the rim of the hole, $(2\pi)^{-1}\int\varphi E_s ds$, where E_s denotes the electric field tangential to the rim of the hole. Therefore the condition for Maxwell's equations to be satisfied is $E_s=0$. This is an extra condition with regard to the solutions e_x, e_y, h_z. For a circular hole the condition is equivalent to $x'e_y-y'e_x=0$ at the rim. This condition is satisfied in Copson's (and Bethe's) theory for the small circular hole, and Copson states that his approximate solution does not violate (4.13) [p. 113]. For sound diffraction the rigorous form of Babinet's principle was given in the reviewer's thesis reviewed above.] *C. J. Bouwkamp* (Eindhoven).

Leontovich, M. On a theorem in the theory of diffraction and its application to diffraction by a narrow slit of arbitrary length. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **16**, 474–479 (1946). (Russian. English summary)

The paper discusses the diffraction of plane electromagnetic waves through an opening in a thin plane screen of perfect conductivity. It is first shown, by an argument essentially due to Rayleigh [Scientific Papers, vol. 4, Cambridge University Press, 1903, p. 324] that the problem can be strictly reduced to that of the diffraction of plane waves by a thin plane lamina, of infinite magnetic permeability, agreeing in size, shape and position with the screen-aperture in the original problem. This is transformed, on replacing the electric and magnetic vectors E, H by H, $-E$, respectively, into the problem of diffraction by a thin perfectly conducting lamina. The results of an investigation by the author of this last problem in the special case where the lamina is a thin strip are then applied to give corresponding results concerning diffraction through a thin slit. Finally, results of the author and M. Levin [Zhurnal Tech. Fiz. **14**, 481–506 (1944)] are applied to show that, with normally incident waves, there is a resonance effect, resulting in the passing of a larger amount of energy, when the ratio of slit-length to wavelength is slightly less than half an odd integer.

E. H. Linfoot (Bristol).

Savornin, Jean. Théorie de la diffraction par une fente à bords métalliques. Cahiers de Physique no. **5**, 21–26 (1941).

Savornin, Jean, et Laurentjoye, R. Contribution à la théorie électromagnétique de la diffraction par une fente. Rev. Optique **25**, 73–81 (1946).

The Fresnel-Kirchhoff theory of diffraction at a slit in a plane screen is unsatisfactory physically since it assumes that the screen is perfectly absorbing, which is far from corresponding to the reflecting properties of a metallic screen. Schwarzschild [Math. Ann. **55**, 177–247 (1902)] showed that the problem of diffraction of plane waves incident normally on a slit in a perfectly reflecting plane screen is equivalent to that of solving a pair of simultaneous integral equations. He did not give a solution in closed form, but obtained a first approximation equivalent to adding the Sommerfeld solutions for two coplanar perfectly reflecting half-planes.

In the first paper under review, the slit is treated as the space between two wedges with parallel edges. It is asserted that the diffraction effect is due to the interference of the waves diffracted by the two wedges and that the solution is given by adding the known solutions of the wedge problem. To take account of the imperfections of the reflecting power of the wedges, the diffracted wave functions are multiplied by a complex constant in the manner suggested by Raman and Krishnan [Proc. Roy. Soc. London. Ser. A. **116**, 254–267 (1927)]. The second paper elaborates Schwarzschild's first approximation from the physicist's point of view.

It is claimed that the results agree well with experiment, but there is little of mathematical interest in either paper.

E. T. Copson (Dundee).

***Zworykin, V. K., Morton, G. A., Ramberg, E. G., Hillier, J., and Vance, A. W. Electron Optics and the Electron Microscope.** John Wiley & Sons, Inc., New York, 1945. xi+766 pp. $10.00.

The first part of the book treats the experimental and descriptive side of electron optics; the second part treats the theoretical side. The opening chapter presents a brief discussion of the properties of the electron and a short description of the motion of the electron in electric and magnetic fields. The geometrical optical properties of the field are illustrated. In chapter II the authors discuss various types of instruments based on the idea of the electron optical properties of electric and magnetic fields. Chapter III contains a review of the electron microscope and many types are described. Chapter IV and V give a description of the parts which enter into the construction of the electron microscope and its alignment and manipulation.

In chapter VI one finds a brief account of the errors involved and the necessary steps and precautions which have to be taken to minimize them. Chapters VII and VIII discuss the regulation voltages employed in the operation of the electron microscope, the type of apparatus used and

the preparation of specimens. In chapter IX the authors discuss the applications of the instruments. In all, part one covers 342 pages.

Part two, the theoretical section, contains ten chapters which cover 404 pages. Chapter X discusses the modern view of the wave nature of the electron. There is a brief discussion of Maupertuis' principle for corpuscles moving in potential electric and magnetic fields and the analogous principle for light rays, namely, the Fermat principle. Chapter XI gives a detailed account of potential fields (electric) and, in particular, contains a good many standard electrostatic problems in two dimensions.

In chapter XII, the authors derive the equations of the trajectories of the electron in electric and magnetic fields by employing the Lagrange equations of motion. After obtaining the equations of the paths, the authors integrate them for particular electric fields described in the preceding chapter. They also give an account of numerical and graphical methods of integration. A short description of the rubber model is given at the end of the chapter. In chapter XIII the authors treat the important case of first order optics, or the so-called Gaussian dioptrics, for axially symmetrical electron optical systems. Their method of procedure is to develop the coordinates of the image in terms of the coordinates of the object and the exit pupil plane by means of power series and then retain only the linear or first order terms of the expansion. Following geometrical optics closely, they obtain the principal optical quantities (focal length, position of the cardinal planes, magnification, etc.) for different types of electrostatic lenses, for instance, the thin lens, the immersion lens, the cathode lens, etc. The chapter concludes with a description of the electron mirror.

Chapter XIV contains an account of magnetic fields in free space and in the neighborhood of iron boundaries which one usually encounters in the construction of the magnetic lenses for an electron microscope. Starting with the fundamental equations of the magnetic field, the authors derive formulae for both the magnetic field H and the vector potential A due to a straight infinite wire carrying a current, a circular wire loop and an axially symmetric field whose value on the axis of symmetry is given. The treatment of the magnetic field distribution near iron boundaries is too brief to be entirely satisfactory. The chapter ends with a description of the apparatus commonly employed for measuring magnetic field distributions. In chapter XV, magnetic lenses are considered and analyzed by obtaining the equations of the trajectory of the electron in a magnetic field which were treated in the preceding chapter. This chapter offers a brief but clear exposition of the motion of an electron in a uniform (homogeneous) field. Solutions are obtained for inhomogeneous fields, in particular, for fields of the type $H=H_0/l+(Z/a)^2$ (H_0=constant), which were treated extensively by Glaser and his associates [see Glaser and Lammel, Arch. Elektrotechnik **37**, 347–356 (1943); these Rev. 7, 398] and by R. G. E. Hutter [J. Appl. Phys. 16, 678–699 (1945); these Rev. 7, 399]. There follows a short account of superimposed electric and magnetic fields as applied to the electron multiplier and the magnetron. The work of Brillouin in connection with the study of the magnetron is not mentioned. Finally, the combined lens (electromagnetic lens) is treated briefly.

Chapter XVI treats the important problem of image errors or aberrations of an electromagnetic optical system. The authors attack the problem by means of Hamilton's point characteristic, or the point eikonal. They follow closely the paper of Rogowski [Arch. Elektrotechnik **31**, 555–593 (1937)] for axially symmetric systems. They derive the eight image errors, five due to the presence of the electrostatic field and three new ones which are due to the presence of the magnetic field. These are given explicitly in terms of the electric and magnetic potentials. One finds also Scherzer's aberration formulae derived directly from the differential equation by means of successive approximations. There is a discussion of the separate aberrations for an electron optical system and a discussion of the aberrations of the electron mirror and the chromatic aberrations of electron lenses and mirrors. The chapter ends with a detailed account of the aberrations of cathode lenses and a brief discussion on space charge effects.

In chapter XVII the authors first consider spherical aberration for different Gaussian types of potential fields and investigate the possibilities of correcting and minimizing these errors by varying the dimensions of the electron lens, aperture and the potential distributions. Next, they consider the error due to coma, curvature of the field and astigmatism and finally they present a short account of the distortion and chromatic aberration effects on the image and also the space charge effect. Chapter XVIII is devoted to high energy or high velocity particles (high voltage) where the relativistic correction of mass must be taken into account in the equations of motion. Electrostatic lenses are briefly considered and a relation is obtained for the relativistic aberration along the axis. There follows a brief treatment of combined lenses; expressions for the chromatic aberration are given. The last chapter discusses the factors which affect the formation of the image, effects such as the scattering and absorption of the electrons by the object due to collision processes, diffraction effects and other structural effects of the objects. The chapter ends with a general discussion of the limit of resolution of the various types of electron microscopes.

Two appendices are added, one on noise in the amplification apparatus and in the scanning microscope and the other presenting the fundamental physical constants and units. An extensive author and subject index closes the book.

The reviewer found a very small number of misprints and misquotations in spite of the length of the book. In his opinion, the book is a valuable contribution to the literature of electron optics not only because it is the only complete book on the subject in any language, but also for its great number of illustrations and wealth of material. The book should be of interest not only to the physicist, but also to the engineer who is interested in electronics and also to the applied mathematician. *N. Chako* (Manhattan, Kan.).

Flamm, L. Der Mechanismus des elektrischen Feldes. Österreich. Ing.-Arch. 1, 105–117 (1946).

This paper gives a mechanical theory of the electromagnetic field. By considering the mechanical motion of electric lines of force, the author develops field equations for regions sufficiently removed from any charge. The results agree with those of Maxwell for charge-free space. It is indicated how the theory can be used to derive the general electromagnetic equations. *C. Kikuchi* (East Lansing, Mich.).

Reulos, René. Les équations de Maxwell et les séries de tourbillons. Cahiers de Physique no. 3, 1–14 (1941).

This paper is a sequel to an earlier work of the author [Ann. Physique (11) 7, 700–789 (1937)] in which it was

shown that a solution of Maxwell's equation can be written in the series form $\mathbf{P}=\sum_{n=0}^{\infty}\mathbf{P}_n$, such that $\mathbf{P}_n=\operatorname{curl}\mathbf{P}_{n+1}$ and $\operatorname{div}\mathbf{P}_n=0$, where $\lambda=(i/c)\partial/\partial t$ and $\mathbf{P}=\mathbf{E}+i\mathbf{H}$. The solution then breaks up into two convergent series, of which one accounts for the ordinary properties of the electromagnetic field, while the other may be designated as the radiative solution. It is to be noted that É. Durand [C. R. Acad. Sci. Paris 219, 510–513 (1944); these Rev. 7, 401] has shown that the classical solution of Maxwell's equation is equivalent to the one given by the author.

C. Kikuchi (East Lansing, Mich.).

Gambarana, Rita. Sopra le condizioni di Love per un'onda elettromagnetica in un mezzo anisotropo. Rend. Sem. Mat. Univ. Padova 13, 5–8 (1942).

Belinfante, F. J. On the longitudinal and the transversal delta-function, with some applications. Physica 12, 1–16 (1946). (English. Esperanto summary)

In the present paper the author introduces the functions, "longitudinal" and "transverse" δ-tensors, defined by

$$\delta_{ij}^{\text{long}}(\mathbf{x})=(2\pi)^{-3}\int d\mathbf{k}(k_ik_j/\mathbf{k}^2)e^{i\mathbf{k}\cdot\mathbf{x}},$$

$$\delta_{ij}^{\text{tr}}(\mathbf{x})=\delta_{ij}\delta(\mathbf{x})-\delta_{ij}^{\text{long}}(\mathbf{x}),$$

so that

$$\mathbf{A}^{\parallel}(\mathbf{x})=\int d\mathbf{x}'\mathbf{A}(\mathbf{x}')\cdot\delta^{\text{long}}(\mathbf{x}-\mathbf{x}'),$$

$$\mathbf{A}^{\perp}(\mathbf{x})=\int d\mathbf{x}'\mathbf{A}(\mathbf{x}')\cdot\delta^{\text{tr}}(\mathbf{x}-\mathbf{x}').$$

Applications to electrostatics, quantum electrodynamics and nuclear field theory are discussed. It is shown that some calculations can be simplified. *C. Kikuchi.*

Ferraro, V. C. A. The induction of currents in infinite plane current-sheets. II. Proc. London Math. Soc. (2) 49, 77–98 (1946).

The first part of this paper is concerned with the distribution of the currents and charges in an infinite plane sheet of finite resistance under the influence of an external electromagnetic field which is arbitrary except for necessary conditions at space infinity and $t=-\infty$. (In the preceding paper [same Proc. (2) 46, 99–112 (1940); these Rev. 1, 224], the special case of a uniformly moving system of external magnetic charges was treated.) The solution is first derived for the special "aperiodic" case in which the inducing system ceases to act at a given instant. It is obtained in terms of two arbitrary functions referring, respectively, to the vector and scalar potential components, and it agrees with the corresponding approximate result in Maxwell's treatise upon neglect of the former. The solution can be interpreted physically, following Maxwell, in terms of equivalent motions of the sheet in its initial state, with velocities $w=R/2\pi$ and $w'=c^2/w$, where R is the resistance per unit area in e.m.u. Second, the general solution is derived in terms of surface and space integrals over derivatives of the potentials of the exciting system. Here interpretation in terms of trailing images is complicated save for the case of Maxwell's approximation. Application is made to the special cases of the dielectric polarization produced by a uniformly moving magnetic pole in a perfectly conducting sheet (neglected by Maxwell) and of a purely electrostatic system in uniform motion. The second part is devoted to the case of infinite plane current sheets ($R=0$) which is of importance in geomagnetism dealing with neutral ionized gases of very low density and in London's theory of superconductors. Important results are the equation of motion in the sheet: $m\mathbf{v}-e/c\mathbf{A}=\operatorname{grad}\psi$, where ψ is an arbitrary space-time function, and the constitutive equation $\sigma=-ne^2\mathbf{A}/mc^2$ for the conductivity, obtained without special assumptions.

H. G. Baerwald (Cleveland, Ohio).

Pollaczek, Félix. Le potentiel du condensateur plan à tube cylindrique superposé. Cahiers de Physique no. 14, 18–34 (1943).

Pollaczek, Félix. Le potentiel du condensateur plan à tube cylindrique superposé. Cahiers de Physique no. 16, 20–32 (1943).

The geometric region studied is an infinite parallel plate region $0\leqq z\leqq l$, $0\leqq r\leqq\infty$ coupled to a semi-infinite circular tube, where r and z are the radial and axial coordinates of a cylindrical coordinate system. It is desired to find the axially symmetric solution of Laplace's equation for this region subject to the boundary conditions that one of the parallel plates ($z=0$) is maintained at unit potential, while the other parallel plate ($z=l$, $r\geqq a$) and the cylinder ($r\leqq a$, $z\geqq l$) are maintained at zero potential. The boundary value problem is formulated with the aid of an appropriate Green's function for each of these regions. The author applies a method of successive approximations to find the potential function. Numerical examples are discussed. It is remarked that the problem has some interest in electron optics.

A. E. Heins (Pittsburgh, Pa.).

Rogowski, W. Bemerkung zur Gegeninduktivität koaxialer Kreisringe. Arch. Elektrotechnik 35, 752–755 (1941).

The mutual inductance of two co-axial circular loops of thin wire is proportional to $\sigma=2k^{-1}(K-E)-kK$, where K and E are the complete elliptic integrals of modulus k, $k^2=4Dd/\{(D-d)^2+4A^2\}$, D, d are the diameters of the two circles and A is the distance of the planes of these two circles. The author expands σ in powers of k in order to find an approximation valid when the distance of the two planes is large compared to the diameters of the loops. The first approximation is derived directly from the physical problem, without making use of the exact solution; the exact expression for σ is used in order to derive better approximations, in particular $\frac{1}{4}k^3\pi/(4-3k^2)$, which is said to be a good approximation for σ in the range $0\leqq k^2<0.8$.

A. Erdélyi (Edinburgh).

Guirinsky, N. K. Le potentiel complexe d'un courant à surface libre dans une couche relativement mince pour $k=f(z)$. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 341–342 (1946).

Eliezer, C. Jayaratnam. The classical equations of motion on an electron. Proc. Cambridge Philos. Soc. 42, 278–286 (1946).

Dirac has shown [Proc. Roy. Soc. London. Ser. A. 167, 148–169 (1938)] that the laws of conservation of energy, momentum and angular momentum lead to the following equations of motion for a charged particle in an electromagnetic field:

$$\text{(1)}\qquad \tfrac{2}{3}e^2(\ddot{v}_\mu+(\dot{v})^2v_\mu)+ev^\nu F_{\mu\nu}=B_\mu,\qquad \mu=0,1,2,3,$$

where $v_\mu=\dot{x}_\mu$, x_μ are the space-time coordinates of the particle, the dots refer to differentiation with respect to proper

time and $F_{\mu\nu}$ are the components of the external electromagnetic field tensor; B_μ must satisfy $v^\mu B_\mu = 0$ and $v_\lambda B_\mu - v_\mu B_\lambda$ must be a perfect differential. In Dirac's paper, the assumption was made that (2) $B_\mu = mv_\mu$. The resulting theory has been shown to have consequences which are not in conformity with our usual ideas of mechanics.

The author replaces (2) by

$$B_\mu = mv_\mu + k(v_\mu(\dot{v})^2 + \tfrac{2}{3}\ddot{v}_\mu) \tag{3}$$

and solves (1) for some special problems. The constant k is written as $k = \tau e^4/m$ and it is shown that if $\tau > 0$ self-acceleration of a free electron is possible as in the original Dirac theory. However, if $\tau < 0$ solutions exist which do not show self-acceleration.

Another problem treated is that of an electron disturbed by a pulse of radiation and it is shown that the solution is in a more symmetrical form than that given by the solution of (1) and (2). Finally, it is shown that for the hydrogen atom equations (1) and (3) allow the particles to collide, in contrast to (1) and (2) which, as the author previously showed, predict that the particles will ultimately recede [Proc. Cambridge Philos. Soc. 39, 173–180 (1943); these Rev. 5, 54]. *A. H. Taub* (Seattle, Wash.).

Markov, M. On the back action of the electromagnetic field on a moving electron. Acad. Sci. USSR. J. Phys. 10, 159–166 (1946).

The author discusses some one-dimensional solutions of the classical system of equations describing the motion of an extended electron subject to the influence of its own and an external electromagnetic field. The basic equations used are those due to Lorentz. In these equations the electromagnetic fields due to the charge and current density are eliminated by using the results of Belousov [Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fis. 9, 658–669 (1939)], who has given a general solution of the Maxwell field equations for arbitrary charges and current densities in terms of Fourier integrals. When this is done one is left with a system of integral equations for the acceleration vector of the center of gravity of the electron. These equations are discussed in the one-dimensional nonrelativistic case, where they reduce to

$$m\ddot{x} = -(4/3)\pi e^2/c^2 \int_0^\infty \int_{-\infty}^t \ddot{x}(s)k \sin kc(t-s) f^2(k)\, ds\, dk + F,$$

where m is the inertial mass of the electron, e its charge, c the velocity of light, $\ddot{x}$ the acceleration of the electron and F the external force. The function $f(k)$ is determined in terms of the charge density by $f(k) = \int \rho(r) e^{ik\cdot r} dv$. The Dirac modification of the classical Lorentz theory of the electron [Proc. Roy. Soc. London. Ser. A. 167, 148–169 (1938); 180, 1–40 (1942); Communications Dublin Inst. Advanced Studies. Ser. A, no. 1 (1943); these Rev. 5, 277; 7, 100] is included in this treatment by setting $f^2(k) = \cos k\lambda$.

The author then discusses various methods for handling the improper integrals involved and claims that solutions of the form $Be^{\alpha t}$ cannot be obtained if the process of taking the limit $\lambda \to 0$ is employed at the proper time, that is, after the integral equation is solved. The argument leading to this assertion is not convincing. *A. H. Taub.*

Markov, M. On the back action of the electromagnetic field on a moving electron. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 16, 800–810 (1946). (Russian. English summary)

An English translation is reviewed above.

Cerrillo, Manuel. Electromagnetic radiation of charged spaces with variable charges. Comisión Impulsora y Coordinadora de la Investigación Científica. (Mexico). Anuario 1944, 101–119 (1945). (Spanish)

The author considers a bounded region in space which is occupied by a continuous distribution of electric charges and currents, the charge and current densities at each point varying sinusoidally in time. He calculates the electromagnetic waves radiated from the region. *L. A. MacColl.*

Fränz, Kurt. Das Reaktanztheorem für beliebige Hohlräume. Elektr. Nachr. Techn. 21, 8–12 (1944).

The author generalizes Foster's reactance theorem from the case of lumped-element reactive networks to lossless resonant cavities. Defining the cavity reactance as the ratio of voltage to current at the input, he derives a formula for this reactance as a function of the frequency ω, namely

$$Z^{\pm 1} = j\omega H \prod_1^\infty (1 - \omega^2/\omega_{2n}^2)/(1 - \omega^2/\omega_{2n-1}^2)$$

which differs from Foster's formula only in being an infinite product instead of a finite product. Here $Z(\omega)$ is a meromorphic function instead of a rational function, but its poles and zeros are still simple and still separate each other. The infinite product actually converges. It is not shown, however, that every properly formed reactance function of this type can be realized by an actual cavity.

O. Frink (State College, Pa.).

Kahan, Théo. Perturbations et pression de radiation dans les cavités électromagnétiques. C. R. Acad. Sci. Paris 223, 785–786 (1946).

Rytov, S. M. Excitation of a hollow spherical resonator by a dipole placed at its centre. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 111–114 (1946).

✠de Broglie, Louis. Problèmes de Propagations Guidées des Ondes Électromagnétiques. Gauthier-Villars et Cie., Paris, 1941. vi+114 pp.

In the opening chapter the author derives Poynting's formula. By applying it to the case of plane waves, he derives formulas both for the total energy and the flux of such an electromagnetic field. After the scalar and the vector quantities (V, A) are introduced satisfying the wave equation and the Lorentz condition, the author introduces two new quantities V' and A' which he calls antipotentials. The fields E and H are expressed in terms of V' and A' as follows: $E = \operatorname{curl} A'$, $H = \operatorname{grad} V' + c^{-1}\partial A/\partial t$. They satisfy the same type of equations and conditions as V and A. Next the Hertzian vector Π is introduced along with its antivector Π' and E, H, V, A, V' and A' are expressed in terms of Π and Π'. Both Π and Π' satisfy the same wave equation. Finally Maxwell's equations are written for a special but important metric, namely: $ds^2 = e_1^2 dx_1^2 + e_2^2 dx_2^2 + e_3^2 dx_3^2$, where $e_1 = 1$ and $e_2/e_3 = f(x_2, x_3)$, thus leading to a new function U introduced by Borgnis [Ann. Physik (5) 35, 359–384 (1939)] which satisfies the differential equation

$$\left\{ \frac{\partial^2}{\partial x_1^2} + \frac{1}{e_2 e_3}\left[\frac{\partial}{\partial x_2}\left(\frac{1}{f(x_2, x_3)} \frac{\partial}{\partial x_2} \right) + \frac{\partial}{\partial x_3}\left(f(x_2, x_3) \frac{\partial}{\partial x_3} \right) \right] + d^2 \right\} U = 0.$$

Electric and magnetic waves are defined by assuming $H_1=0$ and $E_1=0$, respectively. The fields E and H are then expressed in terms of U and its antivector U' for the two types of waves.

In chapter II the author considers wave guides of rectangular section, obtains U in terms of trigonometric functions and then calculates the minimum frequency of propagation inside the guide. Next cylindrical wave guides of circular section are treated and U is expressed in terms of Bessel functions. The characteristic frequencies of the waves capable of being propagated inside the guide are obtained from the roots of these functions. A brief discussion of the superposition of plane waves leads to the formula which expresses Bessel functions in integral form, a special case of Whittaker's formula [Math. Ann. **57**, 333–355 (1903)]. There follows a brief exposition of waves in a coaxial cable, represented by a combination of Bessel and Neumann functions. Guides of elliptic cross section are studied briefly following Brillouin's method, by introducing elliptic coordinates. The Borgnis function U is expressed by Mathieu functions. A short account of the relation between phase and group velocity follows.

A concise treatment of transitory phenomena follows. By developing the fields in orthogonal functions with coefficient $c_n(t)$ varying in time one obtains an equation for the c_n. For periodic forces, $c_n(t)$ contains both periodic and aperiodic vibrations; the energy is calculated. Finally the author derives the second uncertainty relation of Heisenberg, $\Delta t\Delta v \geqq (2\pi)^{-1}$.

Chapter III opens with a discussion of the characteristic vibrations of cavities. The parallelepiped is treated briefly, then followed by a concise analysis of the circular cylinder, where the two types of waves are represented by Bessel functions. Next, one finds a short discussion of a torus of rectangular or circular cross section. The chapter ends with a clear exposition of the vibrations of a sphere.

In chapter IV the author introduces a finite conductivity of the metallic wall. Starting from Maxwell's equations, he calculates the resulting losses for both types of waves of zero and first order in cylindrical wave guides; mention is made of the procedure for higher order waves due to Cotte. The chapter is concluded by a short treatment of the dielectric cable. In chapter V the author studies different types of electromagnetic horns and reflectors, for example, the circular cone, the biconical horn and the sectorial cone. No reference is made to the work of Buchholz on the conical horn [Ann. Physik (5) **37**, 173–225 (1940); these Rev. **1**, 350]. Finally the parabolic mirror is treated.

The last chapter takes up the diffraction of electromagnetic waves by applying Kirchhoff's formula in obtaining Huyghens' principle. Since both the electric and magnetic fields must satisfy Maxwell's equations and also the boundary conditions, one has to extend Kirchhoff's formula to vector quantities. The author follows the analysis made by Kottler [Ann. Physik (4) **71**, 457–508 (1923)]. He applies Kottler's results to the diffraction of electromagnetic waves in guides of rectangular cross section and concludes by presenting a short account of Darbord's results on the radiation from a paraboloidal mirror. The short bibliography and a table of constants close the book. *N. Chako.*

Jouguet, Marc. Des effets d'une discontinuité de courbure sur la propagation dans les guides. C. R. Acad. Sci. Paris **223**, 474–475 (1946).

Pekeris, C. L. Perturbation theory of the normal modes for an exponential M-curve in non-standard propagation of microwaves. J. Appl. Phys. **17**, 678–684 (1946).

In this paper a perturbation method is developed for treating nonstandard propagation of microwaves beyond the horizon in the case when the deviation of the M-curve from the standard (the M-anomaly) can be represented by a term $\alpha e^{-\lambda z}$, where z denotes height in natural units. Here M denotes the modified index of refraction of the air. The procedure followed is to express the height-gain function $U_k(z)$ of the kth mode in the nonstandard case as a linear combination of the height-gain functions $U_m^0(z)$ of all the modes in the standard case: $U_k(z)=\sum_{m=1}^{\infty}A_{km}U_m^0(z)$. The execution of this plan hinges on the possibility of evaluating the quantities

$$\beta_{nm}(\lambda)=\int_0^\infty U_n^0(z)\,U_m^0(z)e^{-\lambda z}dz.$$

It is shown that $\beta_{nm}(\lambda)$ satisfies the equation

$$d\beta_{nm}/d\lambda=(2\lambda)^{-1}+\beta_{nm}(\lambda)(-(2\lambda)^{-1}+A+\tfrac{1}{4}\lambda^2+B^2/\lambda^2),$$

where A and B are linear functions of the characteristic values of the mth and nth modes in the standard case. For large λ the following asymptotic formula is obtained:

$$\beta_{nm}=-\frac{2}{\lambda^2+4\lambda A-2+4\lambda^{-1}B^2}+\frac{8(3\lambda^3+4\lambda A-4\lambda^{-1}B^2)}{(\lambda^2+4\lambda A-2+4\lambda^{-1}B^2)^3}.$$

The characteristic values are to be found from an infinite system of linear equations. For this purpose a simple iterative procedure has been developed, which is rapidly convergent. *S. C. van Veen* (Delft).

Robin, Louis. La propagation d'ondes électromagnétiques dans deux ou plusieurs milieux successifs et la diffraction de ces ondes ramenées à l'étude de problèmes de Cauchy. Revue Sci. **84**, 7–14 (1946).

This is a summary of the author's thesis. The main results were announced in C. R. Acad. Sci. Paris **218**, 135–136, 989–990 (1944); these Rev. **7**, 177, 400.

***Pidduck, F. B. Currents in Aerials and High-Frequency Networks.** Oxford University Press, 1946. iv+97 pp. $2.50.

The basis of the theory developed in this book is a paper by H. C. Pocklington [Proc. Cambridge Philos. Soc. **9**, 324–332 (1897)] in which the propagation of electromagnetic oscillations along wires is discussed; the wire is regarded as a line of singularities along which are distributed electric dipoles tangential to the line. The condition that the electric intensity shall be at right angles to the wire gives an equation for the distribution of the current. The author uses Pocklington's ideas to give a new derivation of F. H. Murray's theory [Amer. J. Math. **53**, 873–890 (1931)]. If $I^\mu(p)e^{i\omega t}$ is the current in the μth aerial of an array, at distance p from a fixed point of that aerial (the distance being measured along the aerial), he obtains a system of integral equations for $I^1(p)$, $I^2(p)$, $\cdots$. These integral equations are deduced under the assumption of a complete skin effect, that is, assuming that the currents flow entirely on the surface. The cross-section of the wires is assumed to be a small circle.

For an open aerial the current vanishes at both ends and can be expanded in a Fourier series,

$$I^\mu(p)=\sum_{m=1}^{\infty}c_m^\mu\{\exp[(im\pi/2l^\mu)p]-(-)^m\exp[-(im\pi/2l^\mu)p]\},$$

where $2l^p$ is the length of the aerial and p is measured from the middle. For the coefficients c_m^p there results an infinite system of linear algebraic equations which can be solved by successive approximations. The quantities appearing in this system allow the definition, independently of the magnetic flux, of the self-impedance of an aerial and of the mutual impedance of two aerials. The theory can be adapted to closed circuits and to networks. With the help of this theory the following problems are discussed in detail: jointed aerials, two straight aerials near a resonant length, two parallel aerials, two straight coplanar aerials, current in a straight receiving aerial, currents in short and long wires, Lecher wires and aerials on Lecher wires, resonant aerials containing a small coil or condenser, closed circuits in particular current in a circular loop, the untuned Y feeder, aerials parallel to the earth. In addition there are some remarks on the relation to transmission line theory and on diffraction theory.

In much of the work use is made of the exponential integral function and of the integrals

$$E_n^s(x)=\int_0^\pi J_0(xy)y^se^{iny}dy.$$

There are sections on these functions and also short five-figure tables of Ei $(\pm ix)$ and of $e^{\pm ix}$, and four-figure tables of $E_1^0(x)$ with its derivative and of $E_1^1(x)$. *A. Erdélyi.*

Tetelbaum, S. On some problems of the theory of highly-directive antenna arrays. Acad. Sci. USSR. J. Phys. **10**, 285–292 (1946).

The electromotive force of a plane broadside array in the field of an elementary dipole, taking into account the curvature of the wavefront, is shown to attain a maximum value as the dimensions of the array are increased. The useful power increases with the dimensions to a maximum value and then decreases; in the limit it varies inversely as the array area. If the broadside array is arranged so that it lies in the spherical wavefront, instead of in a plane, the power output is considerably higher. The radiation resistance and the field structure in the vicinity of the focus of the spherical array are also determined. *M. C. Gray.*

Fock, V. The surface current distribution induced by an incident plane wave. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **15**, 693–702 (1945). (Russian. English summary) [MF 15637]

A translation is reviewed below.

Fock, V. The distribution of currents induced by a plane wave on the surface of a conductor. Acad. Sci. USSR. J. Phys. **10**, 130–136 (1946).

If the wave-length of a plane wave, incident upon a perfectly conducting convex body, is small in comparison with the dimensions of the body and its radii of curvature at each point, the currents induced in the conductor can in the first approximation be obtained from the geometrical-optical solution, that is, assuming that there is reflection according to the laws of geometrical optics from the "illuminated" part of the surface and that there is no induced current at all in the "shadow." This solution breaks down in the penumbral region (near the boundary of the geometrical shadow) where it would imply an impossible discontinuous transition. The purpose of the present paper is to discover an approximation valid in this penumbral region.

The author makes it plausible that the induced current at a point of the penumbral region depends only on the incident wave and the principal radii of curvature of the surface of the conductor at the point considered. Hence the current distribution for any surface can be obtained provided that the distribution is known for one sufficiently general surface. The paraboloid of revolution is a suitable surface; the author briefly indicates the solution in this case. It transpires that the current distribution in the penumbral region can be described (approximately) in terms of a single universal function of one variable. The paper includes short tables of the numerical values of this function. *A. Erdélyi* (Edinburgh).

Leontovich, M., and Fock, V. Solution of the problem of propagation of electromagnetic waves along the earth's surface by the method of parabolic equation. Acad. Sci. USSR. J. Phys. **10**, 13–24 (1946).

The method was proposed by Leontovich [Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **8**, 16–22 (1944); these Rev. **6**, 109]. In the first section of the present paper the discussion of propagation of electromagnetic waves over a plane earth is repeated since the considerations in the original paper by Leontovich need some modifications: the result is again the well-known formula of Weyl and van der Pol, which now appears as the exact solution of the (approximate) parabolic partial differential equation with suitable boundary conditions. In the second section the same problem with a spherical earth is discussed and it is shown that the method leads to the approximation recently obtained (in a different manner) by Fock [C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 310–313 (1945); these Rev. **7**, 100].

The paper concludes on a note of (justified) caution. Since the method is based on neglecting some of the derivatives of the highest order in the partial differential equation governing the problem, it is difficult to estimate the quality of the approximation obtained. From this point of view it is encouraging to observe that the results obtained by Leontovich's method for both plane and spherical (homogeneous) earth agree with results obtained by more reliable, if more cumbersome, processes. *A. Erdélyi.*

Leontovich, M., and Fock, V. Solution of the problem of propagation of electromagnetic waves along the earth's surface by the method of parabolic equation. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **16**, 557–573 (1946). (Russian. English summary)

An English translation is reviewed above.

Rydbeck, Olof E. H. On the propagation of radio waves. Trans. Chalmers Univ. Tech. Gothenburg [Chalmers Tekniska Högskolas Handlingar] **1944**, no. 34, 170 pp. (1944). [MF 16819]

The author gives a comprehensive discussion of the theory of the transmission of radio waves round a spherical earth, surrounded by a radially inhomogeneous concentric spherical shell. By way of introduction he first discusses the wave functions for a plane parabolic layer, obtaining reflection and refraction coefficients in terms of parabolic cylinder functions. For the radio problem he reverts to the classical Watson method for long waves, but for medium or short waves prefers the phase integral or component ray method. A wide range of conditions is covered, for both vertically and horizontally polarized waves, and the effect of the

reflecting shell is clearly illustrated. The formulas obtained are too complex to be quoted, though their form can be inferred from the well-known results for the spherical earth in the absence of the surrounding layer. Applications of the theory to the analysis of ionospheric conditions are indicated and a short table of values of various Bessel functions of orders $\pm\frac{1}{3}$, $\pm\frac{2}{3}$ is included. *M. C. Gray.*

Saha, M. N., and Banerjea, B. K. Wave-treatment of propagation of electromagnetic waves in the ionosphere. Indian J. Phys. 19, 159–166 (1945).

The wave-treatment for the propagation of electromagnetic waves in the ionosphere, first given by Hartree [Proc. Cambridge Philos. Soc. 25, 97–120 (1929)] is simplified. It is shown that the wave is split up into three components, as in the Zeeman effect, one of which is ordinary, the other two extraordinary, in agreement with the observation by Toshniwal [Nature 135, 471–472 (1935)] and Harang [Terr. Magnetism 41, 143–160 (1936)]. The solutions of the wave equations are, however, not given. *C. Kikuchi.*

Wolf, Alfred. Electric field of an oscillating dipole on the surface of a two layer earth. Geophysics 11, 518–534 (1946).

Tychonoff, A. N. On the settling of the electric current in a homogeneous conductive half-space. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] 10, 213–231 (1946). (Russian. English summary)

The following problem is considered. A wire connecting the points B and A is situated on the boundary $z=0$ of the homogeneous conductive half-space $z>0$ (earth). A current of the intensity I_0 is conducted through the wire in the moment t_0. The problem is treated as the solution of Maxwell's equations. The solution is expressed by elementary functions and quadratures and is thoroughly accessible for quantitative and qualitative investigations.

From the author's summary.

Elsasser, Walter M. Induction effects in terrestrial magnetism. II. The secular variation. Phys. Rev. (2) 70, 202–212 (1946).

The analysis developed in part I [Phys. Rev. (2) 69, 106–116 (1946); these Rev. 7, 401] is applied to an interpretation of the secular geomagnetic variations. The effects of the dipole and quadrupole terms in the solutions of the field equations given in the earlier paper are considered.

C. Kikuchi (East Lansing, Mich.).

Parodi, Maurice. Propagation sur une ligne électrique sans pertes dont les paramètres linéiques sont des fonctions exponentielles du carré de l'espace; analogie avec la résolution de l'équation de Schrödinger pour l'oscillateur harmonique. J. Phys. Radium (8) 6, 331–332 (1945). [MF 16303]

The author points out that the differential equation for the propagation of an electromagnetic wave of angular frequency ω along a lossless infinite exponential transmission line is identical with the Schrödinger equation for the linear harmonic oscillator. Assuming that the current vanishes at infinity while the voltage becomes infinite, it is shown that only a discrete set of values of ω are allowable. The solutions for voltage and current are found in terms of Hermite polynomials. *O. Frink* (State College, Pa.).

Raymond, François. Remarque sur la propagation d'un signal électromagnétique sur une ligne hétérogène. C. R. Acad. Sci. Paris 222, 1000–1002 (1946). [MF 16394]

The author previously [same C. R. 220, 497–500 (1945); these Rev. 7, 403] developed a method of resolution of the equations of propagation based on the notion of two waves propagated in opposite senses, $+x$ and $-x$, in the line considered. There he obtained equations

$$(1)\qquad \frac{dR}{dx}=-yR-\frac{1}{2z}\frac{dZ}{dx}(R+S),\qquad \frac{dS}{dx}=+yS-\frac{1}{2z}\frac{dZ}{dx}(R+S),$$

where Z is the impedance and R and S are the Laplace transforms of the currents along $+x$, $-x$, respectively. Based on (1), the coefficients of reflection, at x, for the waves S and R are defined: $d\rho=-\frac{1}{2}dZ/Z$, $d\rho=\frac{1}{2}dZ/Z$. The present note discusses aspects of the coefficient of reflection in the case of a discontinuity in the line. *A. L. Foster.*

Marié, Pierre. Sur une formule rigoureuse du rapport d'atténuation dans un filtre. C. R. Acad. Sci. Paris 222, 869–870 (1946). [MF 16287]

Using the results of M. Parodi on electrospherical polynomials [Memor. Sci. Phys., no. 47, Gauthier-Villars, Paris, 1944; these Rev. 7, 295] the author derives a formula for the attenuation of an electric wave filter composed of n identical four-terminal networks in cascade, in terms of the terminal impedance, the internal impedance of the source, and the iterative impedances of the component networks.

O. Frink (State College, Pa.).

Jaeger, J. C. Switching problems and instantaneous impulses. Philos. Mag. (7) 36, 644–651 (1945). [MF 16981]

The author discusses the theory of transients in lumped passive electric networks caused by switching operations. These problems involve networks started at a time $t=0$ with arbitrarily assigned charges and currents which may or may not be compatible with the charges and currents computed for $t>0$. It is remarked that the problem is intimately related to the theory of impulsive currents and voltages and can be treated with the unilateral Laplace transform. *A. E. Heins* (Pittsburgh, Pa.).

Strecker, F. Die Anwendung der Matrizenrechnung in der Elektrotechnik. Arch. Elektrotechnik 24, 167–175 (1940).

Usunoff, Grigor A. Die Behandlung von Netzwerkaufgaben mittels Matrizen. Arch. Elektrotechnik 36, 115–122 (1942).

Thermodynamics, Statistical Mechanics

Hayes, Wallace D. Transformation groups of the thermodynamic variables. Quart. Appl. Math. 4, 227–232 (1946).

This paper exhibits a group of 32 transformations on E, H, F, G, S, T, P and V which leave the fundamental equations invariant. The quotient group with respect to a normal subgroup of order four gives the octic group found by other investigators. The author avoids the use of absolute values and rules of signs. *C. C. Torrance.*

Henyey, L. G. Near thermodynamic radiative equilibrium. Astrophys. J. 103, 332–350 (1946). [MF 16780]

The radiative equilibrium of an atmosphere in which the excitation conditions depart only slightly from those under strict thermodynamic equilibrium is investigated. The method consists in solving the equations expressing the stationariness of the number of atoms in a given element of state by a perturbation method. More particularly, the number of atoms $N_k(\epsilon)$ in a state k and in an element $(\epsilon, \epsilon+d\epsilon)$ of this state is expressed in the form

$$N_k(\epsilon)=N_k^{(0)}(\epsilon)\{1+\xi_k(\epsilon)\},$$

where

$$N_k^{(0)}(\epsilon)\propto g_k\psi_k(\epsilon)e^{-h\epsilon/kT}=\frac{g_k\gamma_k}{\pi}\frac{e^{-h\epsilon/kT}}{(\epsilon-\epsilon_k)^2+\gamma_k} \tag{1}$$

is the number of atoms under thermodynamic conditions. In (1), g_k is the statistical weight of the state k, ϵ_k the normal position of state k and γ_k its half width. The equations expressing stationariness are solved on the assumption that $\xi_k(\epsilon)$ are small compared to unity and that the exciting radiation departs only slightly from that of a black body at an appropriately chosen reference temperature T.

Allowing suitably for an arbitrary additive constant in the ξ's, it is shown that the equations determining the ξ's can be reduced to the form

$$\xi_k(\epsilon)=\sum_l p_{kl}\xi_l+\phi_k(\epsilon), \tag{2}$$

where p_{kl} denotes the probability that an atom anywhere in the state k will make a transition to the state l and $\phi_k(\epsilon)$ are quantities depending in a somewhat complicated way on the p_{kl}'s and the ψ_l's and on the departures of the exciting radiation from that of the black body.

The solution of equation (2) in its integrated form, $\xi_k=\sum_l p_{kl}\xi_l+\phi_k$, is expressed as $\xi_j=\phi_j+\sum_k q_{jk}\phi_k$, where the matrix $(q_{jk})=\mathbf{Q}$ is related to $\mathbf{P}=(p_{jk})$ according to $\mathbf{Q}=(1-\mathbf{P})^{-1}-1$. With the ξ_k's determined in this way the $\xi_k(\epsilon)$ follow from equation (2). With the problem of population in the various states and in the various elements of states solved in this manner, expressions for the absorption and emission coefficients (including all terms which are of the first order in the ξ's) are obtained and the relevant equations of transfer are formulated.

It should be added that special care is taken to include the transitions to the continuum properly. However, the velocity distribution of the free electron is assumed to be Maxwellian at a temperature T_e, which is later determined from the condition that there is no net exchange of energy between matter and radiation. *S. Chandrasekhar.*

***Hinčin, A. Ya. [Khintchine]. Matematičeskie Osnovaniya Statističeskoĭ Mehaniki. [Mathematical Principles of Statistical Mechanics].** OGIZ, Moscow-Leningrad, 1943. 128 pp. (Russian)

The author shows how to make classical statistical mechanics a respectable rigorous discipline, with a consistent mathematical content. The basic points can be summarized as follows. Inspired by Hamilton's equations and Liouville's theorem on the invariance of phase volume, the standard measure is defined on the energy surface $E=a$. The total measure $\Omega(a)$ of this surface is the derivative with respect to x of the phase volume defined by $E<x$, taken at $x=a$. The function $\Phi(\alpha)=\int_0^\infty e^{-\alpha x}\Omega(x)dx$ and "associated distribution" with probability density $e^{-\alpha x}\Omega(x)/\Phi(\alpha)$, $x\geqq 0$, are fundamental to the development. These depend on the positive parameter α which will be chosen, as explained below, to depend on the energy a assumed by the given system. If the phase space coordinates $x_1, \cdots, x_{2n}$ can be separated into two groups $x_1, \cdots, x_k$, $x_{k+1}, \cdots, x_{2n}$ in such a way that the energy $E(x_1, \cdots, x_{2n})$ can be expressed as a sum of the form $E_1(x_1, \cdots, x_k)+E_2(x_{k+1}, \cdots, x_{2n})$, the two groups of variables are said to determine components of the system. The Φ function and associated distribution can be defined for components (using their own phase space variables) just as for the original system. The fundamental theorem is that the Φ for a sum of components is the product of the individual Φ's and that correspondingly the associated distribution of the sum is the convolution of the component associated distributions. Thus the associated distributions of components combine like the distributions of independent chance variables in the theory of probability. Since the systems considered in statistical mechanics are in effect composed of large numbers of components which are either identical or at worst are divided into a small number of types, the usual apparatus of probability theory, in particular, the central limit theorem, can be applied to replace the involved ad hoc calculations found in the standard texts on statistical mechanics. For example, the function $\Omega(x)$ is put in the form $\Omega(a)=\Phi(\theta)e^{\theta a}\{\frac{1}{2}\pi B+O(n^{-3/2})\}$, where θ is the value of α making $-d(\log\Phi)/d\alpha=a$, the energy value of the given system, and where $B=d^2\log\Phi/d\alpha^2$ (with $\alpha=\theta$); n is the number of components. This value of α is later identified with $1/kT$, where k is Boltzmann's constant and T is the absolute temperature. Using simple approximate expressions of this type, the desired distributions and mean values can be found by surprisingly simple evaluations. For example, the distribution of energy E_1 of a component itself made up of n_1 subcomponents, with $n_1=o(n)$, has approximate density $e^{-\theta x}\Omega(x)/\Phi(\theta)$, where θ is that for the whole system, Φ and Ω that for the component. Thus the associated distribution becomes the approximate energy distribution. This distribution of a small component of a system is the macrocanonical distribution of Gibbs. The distribution of velocity of a monatomic gas is calculated explicitly to derive Maxwell's velocity distribution, the energy equipartition theorem is proved, etc. There is a careful discussion of entropy, designed to show that this quantity is not to be considered a chance variable associated with a complete system, but either a fixed quantity associated with the system (and assigned values of external parameters and temperature) or with a small component of a system, that is to say with a system in heat equilibrium with a thermostat. Since components are not actually independent, because of the required constancy of total energy, the calculation of dispersion of such quantities as the energy of the sum of a large number of components is complicated by component correlations. These are evaluated.

The book is complete in itself, to the point of giving proofs of the ergodic theorem of Birkhoff, with which is joined a careful discussion of the general ergodic problem and of the central limit theorem with remainder. The small number of places where approximation theorems are required make the formal work so simple that the theoretical understanding is not hampered by manipulatory detail.

J. L. Doob (Urbana, Ill.).

Marrot, R. Sur l'équation intégrodifférentielle de Boltzmann. J. Math. Pures Appl. (9) 25, 93–159 (1946).

This extensive study of Boltzmann's integrodifferential equations is undertaken in four chapters. In the first, the classical reasoning leading to the equation is set forth in a

particularly simple form, leading to the symmetry properties of the equation and capable of extension to the case of restricted relativity. (No quantum extensions of the work are indicated.) In the second chapter, Carleman's treatment of Boltzmann's equation in the case of no external forces, together with spherical symmetry of the distribution function in the momentum, is studied with an attempt to generalize the convergence assumptions. In the third, this case is studied in the presence of external forces; these, it is shown, must be conservative for spherical symmetry in the momenta; the initial state of the system must satisfy a certain condition given explicitly. In the fourth chapter, solutions of the integrodifferential equations are considered which are neighboring to the Maxwell distribution. The variation equations are linear and are treated by the methods of Hilbert and Fredholm. *B. O. Koopman.*

Yudin, M. I. Physical mean-forming and the laws of turbulent diffusion. C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 103–106 (1946).

In the statistical description of a mechanical system with a finite number of degrees of freedom, two methods of averaging are employed: (a) average, from $t=0$ to $t=+\infty$, over a given phase line; (b) average, at a given t, over all possible phase lines. The connexion between the two results is given by ergodic theorems. In fluid mechanics, for the study of turbulent flows, it seems that, since the pioneer work of O. Reynolds [Philos. Trans. Roy. Soc. London. Ser. A. 186, 123–164 (1895)], only the method of averaging over time (or space) is employed; however, in this way one is faced with serious difficulties. The author proposes the other way (which he calls "physical mean forming"), that is, to average over a set of states of the fluid satisfying given initial and boundary conditions; he applies his ideas to the diffusion of a cloud of N particles on the z-axis relative to their distribution center. Denoting by $z_K(t)$ and $z_c(t)$ the position of the Kth particle ($K=1, \cdots, N$) and of the center of distribution at time t, the problem is to find the function

$$\sigma^2(t)=N^{-1}\sum\overline{[z_K(t)-z_c(t)]^2}$$

in repeating many times the experiment at the same time t, under identical external conditions, with the same initial conditions $z_c(0)$ and $\sigma(0)$. Assuming the law of turbulent flow structure of Kolmogoroff [same C. R. 30, 301–305 (1941); these Rev. 2, 327] he gets

$$d^2\sigma^2/dt^2=2A\sigma^{\frac{1}{2}}+2N^{-1}\sum\overline{(z_K-z_c)d^2(z_K-z_c)/dt^2};$$

if the second term on the right can be neglected, he finds a diffusion law already proposed by A. M. Obukhov,

$$\sigma^{\frac{1}{2}}(t)-\sigma^{\frac{1}{2}}(0)=Bt.$$

J. Kampé de Fériet (Lille).

Richardson, Lewis F. The probability of encounters between gas molecules. Proc. Roy. Soc. London. Ser. A. 186, 422–431 (1946).

Estimates for the probabilities of the occurrence of binary, ternary, etc. encounters are given. From probability theory the Poisson distribution for the space distribution of molecules is postulated and various physical arguments are used to estimate the duration of encounters.

W. Feller (Ithaca, N. Y.).

Miller, A. R. The number of configurations of molecules on a lattice. Proc. Cambridge Philos. Soc. 42, 303–310 (1946).

The approximation technique of a former paper [same Proc. 39, 54–67 (1943)] is extended to arrangements of chain molecules each of which occupies two or three adjacent sites of a lattice. The resulting partial differential equations satisfy the integrability condition and this is considered as a check on the internal consistency of the method. *F. Zernike* (Groningen).

Auluck, F. C., and Kothari, D. S. Statistical mechanics and the partitions of numbers. Proc. Cambridge Philos. Soc. 42, 272–277 (1946).

The authors consider an assembly of N identical linear oscillators on which n energy quanta $h\nu$ are distributed, either according to Bose-Einstein or Fermi-Dirac statistics, and calculate the thermodynamic functions by well-known formulae, starting from the partition function Z (sum of states). In the Bose-Einstein case Z contains $p_N(n)$, the number of different ways of expressing n as a sum of at most N integers. By an asymptotic calculation, involving Euler's summation formula, the authors find

$$p_N(n)\sim 4^{-1}3^{-\frac{1}{2}}n^{-1}\exp(2^{\frac{1}{2}}3^{-\frac{1}{2}}\pi n^{\frac{1}{2}}), \qquad N\geqq n,$$

which is the asymptotic formula of Hardy and Ramanujan [Proc. London Math. Soc. (2) 17, 75–115 (1918)]. Further results for restricted partitions are also given or announced. The reviewer remarks that no distinction is made between a predetermined number n and the mean value at a given temperature. The mathematical results thus depend on such intricate questions as the equivalence of the canonical and the microcanonical assembly and can only be of heuristic value. *F. Zernike* (Groningen).

BIBLIOGRAPHICAL NOTES

Bulletin de l'École Polytechnique de Jassy.

Vol. 1, no. 1, is dated January–June, 1946. There are to be two numbers a year. The main title is in Romanian: Buletinul Politehnicii "Gh. Asachi" din Iaşi.

Euclides. Revista Mensual de Ciencias Exactas, Físicas, Químicas, Naturales y Aplicaciones Técnicas.

The journal is published in Madrid. Vol. 1, no. 1, was dated March, 1941. The subtitle has varied slightly.

Pozitiva. Revista de Matematici.

Vol. 1, no. 1 appeared in September 1940. There are ten numbers a year. The journal is published in Bucarest.

Simon Stevin. Wis- en Natuurkundig Tijdschrift.

This is a bimonthly, published at Groningen. The first issue is numbered no. 1 of the 25th year (1946–47); Simon Stevin continues the 24th year (12th volume) of Wis- en Natuurkundig Tijdschrift, the 18th year of Christiaan Huygens and the 13th year of Mathematica (Zutphèn) B.